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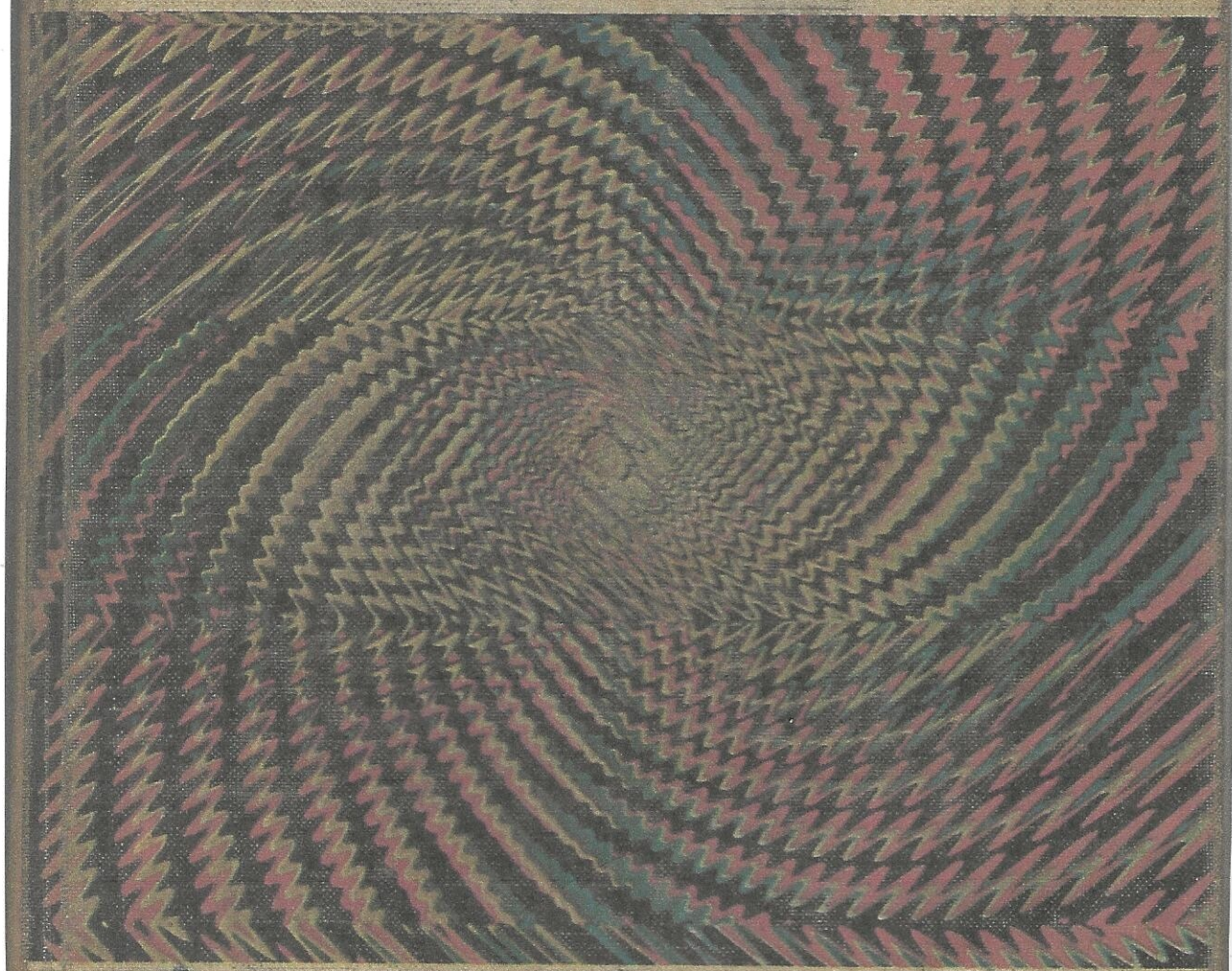

Joseph N. Payne Floyd F. Zamboni Francis G. Lankford, Jr.

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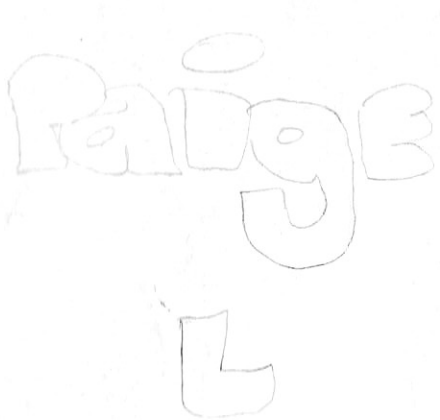

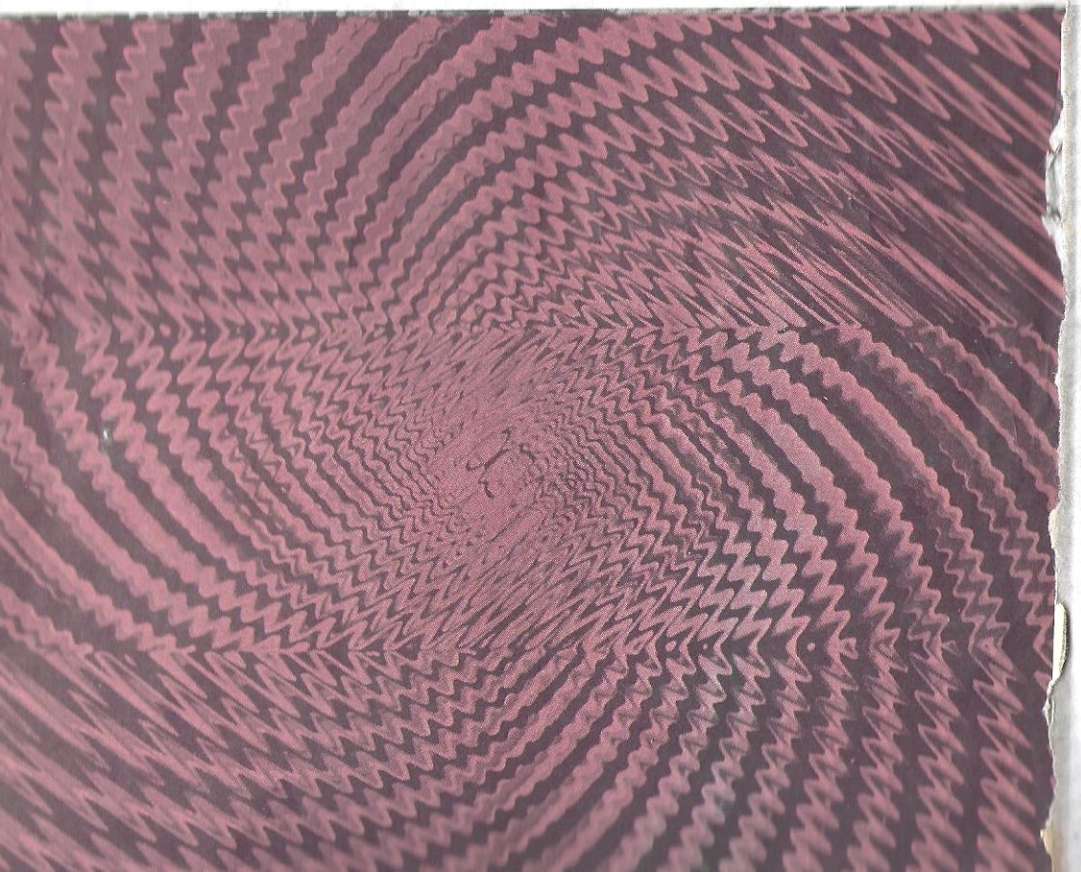

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ABOUT THE COVER

The cover shows a photograph taken by Sheldon S. Rose. He created the moiré pattern using four films of the same basic design, each in a different color. He placed the films one on top of the other and then photographed the result through a textured sheet of glass.

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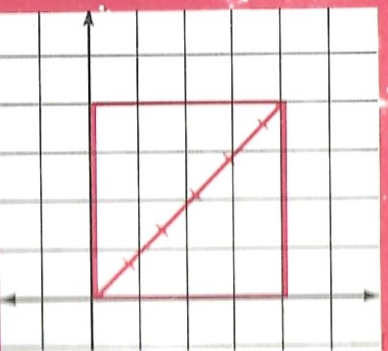


CHAPTER 1
SETS AND NUMBERS

Much of mathematics has been created by man to describe things in his environment. You use mathematics to count and tell the number of things, to describe shapes, to calculate distances and dimensions, to solve problems.

In this chapter you will review the ideas of sets and subsets. You will look at sets of numbers and relations between them, and review the postulates for operating with sets and numbers. Working with equations will be introduced.

For how many of the following questions do you know the answer?

- What do the set of horses and the set of telephone poles in the photographs have in common?
- Name some other sets in the photographs at the left. Does each set have a finite number of elements or an infinite number of elements? Can you count the stars in the center photograph? If you lived long enough, could you count the grains of sand on the beach in the lower right photograph? What is an infinite set?
- What are you doing when you count? How can you tell if two sets have the same number of elements?
- Can you name the part that one telephone pole is in relation to the whole set? Is the answer one of the whole numbers?
- Look at the diagram of the square. Can you count the number of units in the length of each side? Can you tell the length of the diagonal, as marked with the same units?
- Can you find something in one of the photographs that makes you think of a set of points that is a triangle? a line?

When you have studied this chapter, you will know the answers to all these questions and to many more.

1–1 Sets and Elements

Long ago a mathematician and scientist named Galileo Galilei (1564–1646) noticed something very odd. He wrote a list of numbers used for counting, which today are called the **natural numbers** or **counting numbers.**

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots$$

(The three dots mean that the list can be continued in the same way. The natural numbers continue without bound.) Underneath this list, Galileo wrote the product of each number with itself, which is called the **square** of the number. The square of 3, written as 3×3 or 3^2, is 9.

$$1, 2, 3, \ 4, \ 5, \ 6, \ 7, \ 8, \ 9, \ 10, \ 11, \ 12, \cdots$$
$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, \cdots$$

Galileo noticed that the square numbers seem to be farther and farther apart as the numbers get larger. But there seem to be just as many square numbers as there are natural numbers. In fact, for each natural number there is one and only one square number, and for each square number there is one and only one natural number, called a **square root.** The natural-number square root of 16, which is 4, is written as $\sqrt{16}$.

The **even natural numbers** can also be matched, one to one, with the natural numbers.

$$1, 2, 3, 4, \ 5, \ 6, \ 7, \ 8, \ 9, 10, 11, 12, \cdots$$
$$2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, \cdots$$

Do there seem to be as many even numbers as there are natural numbers? Can you show a similar matching for the **odd numbers,** 1, 3, 5, 7, 9, 11, $\cdots$, with the natural numbers?

Galileo did not know how to explain this mystery. Many years later, at the end of the nineteenth century, a mathematician named Georg Cantor investigated the problem thoroughly and, in the process of explaining it, invented a new branch of mathematics that is useful in many ways. Cantor's new mathematics is called **set theory,** and in the beginning of this book, you will learn enough about sets to understand Galileo's mystery.

A **set** is a well-defined collection of objects called **elements.** You think of the collection as a single thing. When you think of the boys and girls in your room as a set of persons, you think of *one* group.

Well-defined means that you can tell if any given object belongs or does not belong to the set, that is, whether or not it is an element of the set. If the set under consideration is the set of green vegetables, then you know *spinach* is an element of the set, but *carrot* is not. The set is well defined.

The idea that an element belongs to a given set is used so often that a special symbol is used to express it.

The symbol $\in$ means is an element of, and $\notin$ means is not an element of.

Are the following two statements correct? The first statement is read "*Spinach* is an element of the set of green vegetables."

Spinach $\in$ The set of green vegetables.
Carrot $\notin$ The set of green vegetables.

Look at the examples in the table below showing the ways that a set may be indicated, or named. The **rule method** denotes the set by using words, formulas, or properties. The **roster method** denotes the set by listing the elements of the set in braces, { }. No matter how a set is denoted, it must be well defined.

	Rule method	Roster method
Numbers	W = The set of whole numbers.	W = {0, 1, 2, 3, $\cdots$}
Points	P = The set of points A, B, and C.	P = {A, B, C}
Teams	T = The set of baseball teams in the American League.	T = {Tigers, Indians, Twins, Orioles, Yankees, Senators, Red Sox, Angels, Athletics, White Sox}
No elements	$\emptyset$ = The empty set, or the null set.	$\emptyset$ = { }

In the examples the letters W, P, and T are names for the sets. In the last example the symbol $\emptyset$ is the name for the **empty set,** the

set that has no members at all. The set of men 17 feet tall is the empty set, or $\{\ \}$. Is the set of purple spots on this page also the empty set?

Both W and $\{0, 1, 2, 3, \cdots\}$ denote the same collection, as is indicated by the equals sign.

Sets are used in geometry as well as in algebra. For instance, a line segment is defined in terms of sets. Line segment AB, $\overline{AB}$, is defined as the set of points A and B and all the points between them. In this case, no braces are used, because $\overline{AB}$ has been defined as a set.

Now try these

— Refer to the examples in the table on page 3, and replace the ● with $\in$ or $\notin$ to make true sentences.

1. 2 ● W
2. B ● P
3. Orioles ● T
4. $6\frac{1}{2}$ ● W
5. $\sqrt{2}$ ● W
6. Line segment AB ● P
7. Triangle ABC ● P
8. Angels ● T
9. 1 ● $\emptyset$
10. Yankees ● $\emptyset$

Answers: **1.** $\in$ **2.** $\in$ **3.** $\in$ **4.** $\notin$ **5.** $\notin$ **6.** $\notin$ **7.** $\notin$ **8.** $\in$ **9.** $\notin$ **10.** $\notin$

Checkpoint

1. What is the difference between the rule and the roster methods for describing a set?
2. What do the symbols $\in$ and $\notin$ mean?
3. What symbol is used to name the empty set?
4. When is a collection a well-defined set?

Exercises

A — The sets W, P, T, and $\emptyset$ used in the sentences below are from the examples in the table on page 3. Read each mathematical sentence and tell whether it is *True* or *False*.

1. $3 \in$ W
2. $2\frac{1}{2} \notin$ W
3. $D \in$ P
4. $C \in$ P
5. White Sox $\notin$ T
6. $97 \in$ W
7. Cardinals $\notin$ T
8. $86\frac{1}{2} \in$ W
9. $a \in \emptyset$
10. $0 \notin \emptyset$

— Write the members of each of the following sets by the roster method. A few elements of some sets are shown.

11. The set of odd whole numbers less than 16.

$\{1, 3, 5, _?_, _?_, _?_, _?_, _?_\}$

12. The squares of all whole numbers less than 6.

$\{0, 1, 4, _?_, _?_, _?_\}$

13. The whole-number square root, $\sqrt{\ }$, of each number in your answer to Exercise 12.

$\{0, 1, 2, _?_, _?_, _?_\}$

14. The natural numbers from 2 through 37 that are exactly divisible by 7.

$\{7, 14, _?_, _?_, _?_\}$

15. The consecutive, odd whole numbers from 30 to 40.

$\{31, 33, _?_, _?_, _?_\}$

16. The set of *prime numbers* less than 20. (A **prime number** is a whole number greater than 1 that has only one pair of whole-number factors, itself and 1.)

$\{2, 3, 5, _?_, _?_, _?_, _?_, _?_\}$

17. The set of all *composite numbers* less than 20. (A **composite number** is a whole number greater than 1 and not prime; that is, it has more than one pair of factors: $6 = 6 \times 1$ or 3×2.)

$\{4, 6, 8, 9, _?_, _?_, _?_, _?_, _?_, _?_\}$

18. The set of line segments connecting the points A, B, and C.

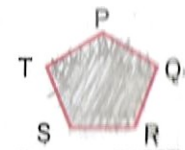
$\{\overline{AB}, \overline{BC}, _?_\}$

19. The set of line segments that are the five sides of pentagon $PQRST$.

$\{\overline{PQ}, \overline{QR}, _?_, _?_, _?_\}$

20. The set of vertices (points of intersection of sides) of pentagon $PQRST$.

$\{P, Q, _?_, _?_, _?_\}$



Pentagon PQRST

21. The set of teachers in your classroom.
22. The set of infants in your classroom.

— Replace each ● with $\in$ or $\notin$ to make true sentences. (A **multiple** of a number is any of the products of the given number and some other whole number.)

N = The set of natural numbers. $\{1, 2, 3, 4, 5, \cdots\}$
W = The set of whole numbers. $\{0, 1, 2, 3, 4, \cdots\}$
E = The set of even whole numbers. $\{0, 2, 4, 6, 8, \cdots\}$
D = The set of odd whole numbers. $\{1, 3, 5, 7, 9, \cdots\}$
M = The set of whole numbers that are multiples of 3. $\{0, 3, 6, 9, 12, \cdots\}$

23. 5 ● N **24.** 7 ● M **25.** 8 ● W
26. 15 ● M **27.** 7 ● E **28.** 198 ● D
29. $8\frac{1}{2}$ ● N **30.** $\sqrt{9}$ ● W **31.** $(2\frac{1}{2} + \frac{1}{2})$ ● W
32. (8×2) ● W **33.** $(8 \div 3)$ ● W **34.** $(16 \div 9)$ ● N
35. $(15 + 6)$ ● N **36.** $(9 + 7)$ ● D **37.** $(12 + 96)$ ● E

B **38.** List the members of the set of all whole numbers less than 20, each of which is named by a two-digit numeral whose tens digit exceeds the units digit by one.

39. What element is in the set $\{0\}$? Does $\emptyset$ have any elements?

C **40.** List the members of the set of whole numbers that are square roots of whole numbers less than 25.

41. List the members of the set of prime numbers between 100 and 200. How many elements does this set have?

1–2 Subsets

Study the following examples that illustrate the meaning of **subset.**

EXAMPLE 1. Think of set P as the set of people in the illustration, G as the set of girls, and B as the set of boys. Name the elements of set P; of set G; of set B. Is each element of G also an element of P?

Since each element of the set of girls is an element of the set of people, the set of girls is a *subset* of the set of people.

Definition A set G is a subset of a set P if and only if every element of set G is an element of set P.

The symbol $\subseteq$ means *is a subset of.* The sentence $G \subseteq P$ is read "G is a subset of P."

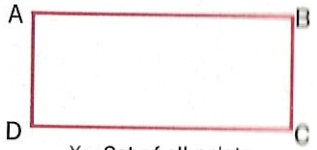
Is $B \subseteq P$ true? Explain, using the definition of subset.

EXAMPLE 2. Here is Galileo's problem, which was stated on page 2, written with the notation of sets.

$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots\}$
$S = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, \cdots\}$

Is every element of S also an element of N? Which is true, $N \subseteq S$ or $S \subseteq N$? Every element of S is an element of N, so $S \subseteq N$ is true.

EXAMPLE 3. Let X be the set of points of the rectangle *ABCD*. Think of the subset of points $\overline{AB}$. Is each point of $\overline{AB}$ an element of X? Then is $\overline{AB} \subseteq X$ true?

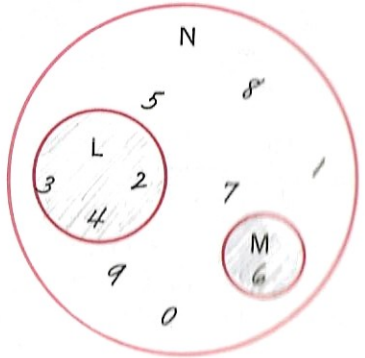


X = Set of all points of rectangle ABCD

Since each point of the segment *AB* is also a point of the rectangle, the segment *AB* is a subset of X, and $\overline{AB} \subseteq X$ is true.

EXAMPLE 4. Name the members of sets N, L, and M. Is each member of L also a member of N? Is each member of M also a member of N?

$N = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$L = \{2, 3, 4\}$ $M = \{6\}$



Each member of L and of M is a member of N.

Tell why these sentences are true.

a. $L \subseteq N$ b. $M \subseteq N$

The symbol for *is not a subset of* is $\not\subseteq$.

Tell why the following are true.

c. $L \not\subseteq M$ d. $N \not\subseteq M$
e. $N \not\subseteq L$ f. $M \not\subseteq L$

Now answer this question. Is each member of set N a member of N? Then, by the definition of subset, the truth of the following statement should be obvious.

Every set is a subset of itself.

Thus, the sentences $P \subseteq P$, $N \subseteq N$, and $X \subseteq X$ are true, where P, N, and X are the sets in Examples 1–3 on pages 6 and 7.

Look at Example 1 again. Suppose you are to choose a subset of P to come to your party. Name the members of three subsets that you might invite. Do you think that one of your subsets could include no person of set P? If you invited no person from set P, you would invite the empty set. If you listed all the possible subsets of people that you invite, it would be sensible to include the empty set. Are there any elements in the empty set that are not in set P? Therefore, you can conclude that the following is true.

The empty set is a subset of every set.

Using the sets from Examples 1–3, these sentences are true: $\emptyset \subseteq P$, $\emptyset \subseteq G$, $\emptyset \subseteq B$, $\emptyset \subseteq X$, $\emptyset \subseteq N$, $\emptyset \subseteq L$, $\emptyset \subseteq M$. The same sentences can be written using braces: $\{\ \} \subseteq P$, $\{\ \} \subseteq G$, and so forth.

Let set A be $\{1, 2, 3\}$. If $B = \{1, 2\}$, is B a subset of A? Does B contain all the elements of A? Then B is a **proper subset** of A.

Definition A proper subset of A is any subset of A except the subset A itself.

The symbol $\subset$ means *is a proper subset of*. The symbol for *is not a proper subset of* is $\not\subset$.

Suppose sets A and B are as shown at the right. Is $A \subseteq B$ a true statement? $B \subseteq A$? Are the members of A exactly the same as the members of B? This leads to the definition of **equal sets.**

$A = \{1, 2, 3, 4\}$
$B = \{4, 3, 1, 2\}$

Definition Set A is equal to set B if and only if A is a subset of B and B is a subset of A.

The definition can be stated in mathematical symbols as follows.

$A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Checkpoint

1. When is set B a subset of set A? When is set B not a subset of set A?
2. Is set A always a subset of A? a proper subset of A?
3. Is the empty set a subset of any set?
4. When is it true that set S is equal to set T?
5. When is set S a proper subset of set T?

Exercises

A — If $A = \{3, 5, 12, 15, 17\}$, list the elements for each of the following subsets of set A.

1. The subset of A whose elements are exactly divisible by 5.
2. The subset of A whose elements are exactly divisible by 3.
3. The subset of A whose elements are exactly divisible by 1.
4. The subset of A whose elements are prime numbers.

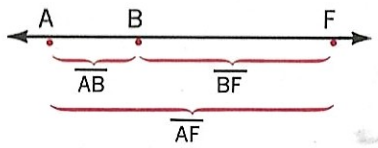
— If $T = \{1, 3, 4, 8, 10, 16\}$, list the elements for each of the following subsets of T.

5. The subset of T consisting of even whole numbers.
6. The subset of T consisting of odd whole numbers named by two-digit numerals.
7. The subset of T consisting of all elements of T that are perfect-square whole numbers. (4 is a perfect square because $2 \times 2 = 4$ is true, but 5 is not a perfect square because 5 is not the square of any whole number.)

— What is the number of elements in each of these sets?

8. The set of natural numbers that divide 36 with zero as a remainder.
9. The set of even natural numbers less than 40.
10. The set of prime numbers between 40 and 50.
11. The set of digits used in our decimal system.

Tell whether each sentence is true or false. (Refer to the diagram below for Exercises 12–16.)



12. $\overline{AB}$ is a subset of $\overline{AF}$.

13. $\overline{AF} \subseteq \overline{AF}$

14. $A \subset \overline{AB}$

15. $\emptyset \subset \overline{AF}$

16. $\overline{AF} \not\subseteq \overline{AB}$

17. The set {rectangle, square, parallelogram, rhombus} is a subset of the set of quadrilaterals.

18. {0} is a subset of the set of natural numbers.

19. {0} is a subset of the set of whole numbers.

20. The set of natural numbers is a subset of the set of whole numbers.

21. For every set A, $A \subseteq A$ is true.

22. For every set A, $\emptyset \subseteq A$ is true.

23. For every set A, $A \subseteq \emptyset$ is true.

Replace ● with $=$ or $\neq$ in each of the following to make the sentence true.

24. A ● B if A = {a, b, c} and B = {a, b}.

25. C ● D if C = {2, 3, 5, 7} and D = {2, 5, 7, 3}.

26. R ● T if $R = \{\frac{1}{2}, \frac{1}{10}, \frac{1}{5}\}$ and T = {0.5, 0.1, 0.2}. (*Hint:* Think of R and T as sets of numbers, not as sets of numerals.)

27. J ● $\emptyset$ if J is the set of natural numbers greater than 5 and less than 6.

28. {0} ● $\emptyset$ (*Hint:* How many members are in {0}? in $\emptyset$?)

Classify the first set of each of the following as either a proper subset of the second set or equal to the second set.

29. {3, 6, 9, 15} and the set of natural numbers between 1 and 18 that are exactly divisible by 3.

30. The set of odd whole numbers less than 10 and {1, 3, 5, 7, 9}.

31. The set of prime numbers less than 12 and {2, 3, 5, 7, 11}.

32. The set of numbers represented by the numerals on a 12-hour clock and {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

33. The set of whole numbers greater than 3 and less than 10, and {4, 5, 6, 7, 8, 9, 10}.

Does a subset relationship exist between X and Y for each of the following? If so, write a sentence using $\subset$ or $\subseteq$.

34. X = {0, 1, 3, 5, 7}; Y = {1, 5, 7}

35. X = {a, b}; Y = {a, b, e}

36. X = the set of all natural numbers less than 20; Y = the set of prime numbers less than 25.

37. X = the set of natural numbers greater than 3 and less than 9; Y = the set of natural numbers less than 8.

Given the following three sets, indicate *True* or *False* for each statement.

A = {1, 3, 5, 7}
B = {1, 7}
C = {1, 3, 5, 7, 9, 11}

38. $B \subseteq C$ **39.** $A \subseteq B$ **40.** $C \subseteq C$

41. $B \subseteq A$ **42.** $C \subseteq A$ **43.** $A \subseteq C$

B Tell whether or not set A is a subset of set B.

44. A = {0, 5} and B = {0}

45. A = $\emptyset$ and B = {0}

46. A is the set of all natural numbers and B is the set of all natural numbers that may be divided evenly by 10.

C **47.** To prove that $\emptyset \subseteq N$ is true for every set N, you reason indirectly. Answer the questions that follow to understand the proof.

a. Either $\emptyset \subseteq N$ or $\emptyset \not\subseteq N$ is true. How do you know this?

b. Suppose $\emptyset \not\subseteq N$ is true. What does this imply? Since N is well defined, you know all the elements in N. Then there is at least one element in $\emptyset$ that is not in N. Why?

c. Does $\emptyset$ have any elements? Why?

d. In **b**, you found that there is an element in $\emptyset$. From **c** you know that $\emptyset$ has no elements. Could both be true?

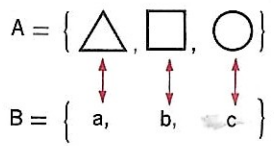
e. Does the supposition that $\emptyset \not\subseteq N$ is true lead to contradictory statements? What are they? Then the supposition $\emptyset \not\subseteq N$ is false.

f. In **a** you saw that either $\emptyset \subseteq N$ or $\emptyset \not\subseteq N$ is true. If $\emptyset \not\subseteq N$ is false, then must $\emptyset \subseteq N$ be true?

This is proof that $\emptyset \subseteq N$ is true for every set N.

1–3 Equivalent Sets and Infinite Sets

Consider sets A and B. Although A and B are not equal sets, they have something in common. Can you match each element of A with just one element in B so that each element of B is matched with exactly one element of A? Then the two sets are in **one-to-one correspondence.**



Definition Two sets, J and K, are in one-to-one correspondence if each element in J is matched with exactly one element in K and if each element in K is matched with exactly one element in J.

In Galileo's problem, set S is a proper subset of set N.

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \cdots\}$$
$$S = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, \cdots\}$$

Do sets N and S appear to have the same number of elements? Are they in one-to-one correspondence? Can you name the number of elements in sets N and S?

Definition Equivalent sets are sets that can be placed in one-to-one correspondence.

From the definition, you can tell that N and S are **equivalent sets** and that A and B are equivalent. If B = {a, b, c} and X = {b, c, a}, are B and X equal sets? Are B and X equivalent sets? Why are all equal sets also equivalent sets?

If R = {0, 6, 27} and Q = {270, 6}, are R and Q equal? Are they equivalent? R and Q are **nonequivalent** sets.

Now look again at the equivalent sets B and X. What number property do the two sets share?

Two equivalent sets have the same number of elements.

Sets B and X each have three elements. You can think of the number 3 as the common property of all sets equivalent to B or X, such as set A. Since N and S are equivalent, they have the same number of elements. But you cannot name a natural number, such as 3, or 6, or 100, to identify that number. The difference between set A and set N is that A is **finite** and N is **infinite.**

Look at some other examples of finite and infinite sets.

Finite sets

a. The set of students in your class.
b. The set of cities in the U.S.
c. The set of molecules in your body.
d. The set of whole numbers, zero through one trillion.
e. The set of grains of sand on the beach.
f. The set of stars in the Milky Way.
g. The set of atoms in the earth.
h. The set of pages in this book.

Infinite sets

i. The set of whole numbers.
j. The set of even numbers.
k. The set of points on a line segment.
l. The set of multiples of ten.
m. The set of prime numbers.
n. The set of composite numbers.
o. The set of lines that contain a given point.
p. The set of numbers named with 1 as the numerator and a natural number as the denominator, that is, $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots$.

Definition A finite set has elements that can be counted, with the counting coming to an end. The empty set is a finite set.

This does not mean that the physical job of counting the elements can be done by any one person or by many persons. It merely means that it is mathematically possible.

Definition All sets that are not finite sets are infinite sets.

It is impossible to count the elements of an infinite set with the counting coming to an end. Look at each infinite set in the list above, and tell why it is infinite. (*Hint:* If there is a new element beyond any given element in a set, the set is infinite.) While you cannot count the elements of an infinite set with the counting coming to an end, you can tell if two infinite sets are equivalent.

Look again at the set of natural numbers and the set of even numbers. The set of even numbers is a proper subset of the set of natural numbers. Are these two sets equal? Are they equivalent? Are they finite or are they infinite?

$$N = \{1, 2, 3, 4, 5, 6, 7, \cdots, n, \cdots\}$$
$$E = \{2, 4, 6, 8, 10, 12, 14, \cdots, 2 \times n, \cdots\}$$

If n represents any natural number, what does $2 \times n$ represent? In these sets what even number matches 5? 7? 25? 1000? any natural number denoted by n? For each natural number, can you name one and only one even number that matches it?

What natural number matches the even number 4? 12? 50? 1,000,000? any even number $2 \times n$? For each even number, can you name one and only one natural number that matches it? Then the sets N and E are equivalent: they have the same number of elements.

Definition The name for the number of elements in the set of natural numbers and in all sets equivalent to the natural numbers is aleph-null. The symbol for the number is $\aleph_0$.

It may seem strange to match the elements of a set with a proper subset of itself. But it can be done with infinite sets. In fact, this matching property can be used to define *infinite sets*. Why can you not place a finite set in one-to-one correspondence with a proper subset of itself?

Checkpoint

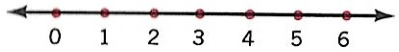
1. When are two sets said to be in one-to-one correspondence?
2. When are two sets equivalent sets?
3. When are two sets nonequivalent sets?
4. What is a finite set?
5. What is an infinite set?
6. When do two sets have the same number of elements?
7. How many elements are in the set of natural numbers?

Exercises

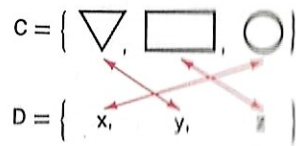
A — Write *Equivalent* or *Nonequivalent* for each pair of sets in Exercises 1–11. Also, identify each set as *Finite* or *Infinite*.

1. $\{1, 2, 3\}$ and $\{6, 7, 8\}$
2. $\{3, 4, 5, 6\}$ and $\{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}\}$
3. The set of even whole numbers between 40 and 50 and the set of even whole numbers between 50 and 60.
4. The set of odd whole numbers less than 25 and the set of prime numbers less than 25.
5. The set of natural numbers that are exact divisors of 36 and the set of digits in the decimal system.
6. The set of vertices of triangle ABC and the set of line segments that form the triangle.

7. The set of vertices of any polygon and the set of sides of the polygon.
8. The set of teachers in your school and the set of adult females in your school.
9. The set of pages in this book and the set of natural numbers 1 through 576.
10. The set of fingers on a normal hand and $\{1, 2, 3, 4, 5\}$.
11. The set of all circles and the set of centers of all circles.
12. The graph shows the set of whole numbers matched with certain points on a line. Is each number matched with exactly one point? Is the given set of points on the line equivalent to the set of whole numbers?



13. The drawing shows one way that sets C and D can be placed in one-to-one correspondence. Make drawings to show five other ways that this can be done.



B 14. Indicate a 1–1 correspondence between the even natural numbers, E, and the odd natural numbers, D. Describe a pattern that indicates how the elements in D are to be matched with the elements in E. What conclusion can you draw about these two sets? How many members has each set?

15. Indicate a 1–1 (one-to-one) correspondence between

$$N = \{1, 2, 3, 4, 5, 6, \cdots\}$$

and

$$B = \{100, 101, 102, 103, 104, 105, 106, \cdots\}.$$

Describe a pattern that indicates how the elements in B can be matched with the elements in N. What can you conclude about these two sets? How many members has each set?

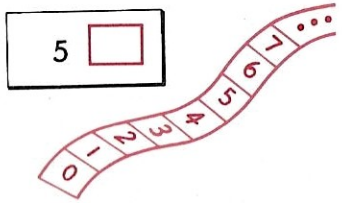
C 16. Is the set of numbers $F = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots, \frac{1}{n}, \cdots\}$ equivalent to the set of natural numbers? How many members has F?

1–4 Variables and Expressions

If you buy a given number of five-cent stamps, the total cost is 5 times the number of stamps. The number of stamps can be any number in the set $W = \{0, 1, 2, 3, \cdots\}$. To represent any number in the set, you can use a symbol such as n, x, or $\square$. Such symbols are called **variables.**

Definition A variable is a symbol (usually a letter) that may represent any element in a given set. The given set is called the replacement set for the variable.

The expressions $3 \times n$, $3 \cdot n$, and $3n$ mean "3 times any number chosen from the replacement set." What would $5\square$ mean? In the diagram at the right, the box, $\square$, is the variable and the numbers shown on the strip form the replacement set.



If $\square$ is replaced by 2, then $5\square$ names 5×2, or 10. Tell the numbers that complete the following table.

Number from replacement set	0	1	2	3	4	5	6	7	⋯	n
$5\square$ is	0	5	10	15	?	?	?	?	⋯	?

The set that results from $5\square$ is the set of multiples of 5.

The replacement set in this example of buying any number of stamps is a finite set of whole numbers. Why would it not make sense to have $6\frac{1}{2}$ in the replacement set? The replacement set is often determined by the nature of the problem; other times, the choice of the replacement set is arbitrary. Sometimes the replacement set is called the *universal set.*

EXAMPLE. Suppose that each table in a furniture store has 4 legs and that n represents the number of tables. Then $4n$ represents the total number of legs. What replacement set is sensible?

The sensible replacement set is $\{0, 1, 2, 3, 4, \cdots, n\}$ because there must be a whole number of tables.

The group of symbols $2 + 5x$ is an **algebraic expression.** An algebraic expression is made up of one or more terms joined by + or −. A **term** is the product and/or quotient of numerals and variables, for example, $\frac{1}{2}xy$, $3mn^2$, $\frac{a}{2b}$. Any one of the factors of the term is a **coefficient** of the other factors. Thus, in the term $\frac{1}{2}xy$, $\frac{1}{2}$ is the coefficient of xy; x is the coefficient of $\frac{1}{2}y$; and y is the coefficient of $\frac{1}{2}x$. But *coefficient* usually means **numerical coefficient.** So when you are asked for the coefficient of an expression such as $3mn^2$, it is 3.

An algebraic expression names a number when the variable is replaced by an element in the replacement set. To find the number named by $2 + 5x$, you need to know whether it means "Add 2 and 5, and then multiply x by the sum" or "Multiply x by 5, and then add the product to 2." Punctuation marks can be used to make the meaning of the expression $2 + 5x$ clear. The most common punctuation marks in algebra are parentheses. Look at the expressions $(2 + 5)x$ and $2 + (5x)$.

The expression $(2 + 5)x$ means "Add 2 and 5, and then multiply x by the sum." Complete this table. If the replacement set were W, what would be the set of values for $(2 + 5)x$?

x	0	1	2	3	4	5	6
$(2+5)x$	0	7	14	21	?	?	?

The expression $2 + (5x)$ means "Multiply x by 5, and then add the product to 2." Complete this table. If the replacement set for x were the set of multiples of 5, what would be the set of values for $2 + (5x)$?

x	0	5	10	15	20	25
$2 + (5x)$	2	27	52	77	?	?

An agreement is usually made on the **order of operation** so that you will be able to write the expressions more easily. Unless parentheses or other grouping symbols show clearly that something else is to be done, the standard agreement on the order of operations is as follows.

First multiply and/or divide in the order that the operations appear; then add and/or subtract in the order that these operations appear.

Thus, for $2 + 5x$, you multiply first, and then add, and no parentheses are necessary.

Note that the set of values for x and the set of values for $2 + 5x$ are equivalent.

$$\{0,\ 5,\ 10,\ 15,\ 20, \cdots\}$$
$$\{2,\ 27,\ 52,\ 77,\ 102, \cdots\}$$

Each value of x is matched with just one value of $2 + 5x$, and each value of $2 + 5x$ is matched with just one value of x.

Now try these

Write a single numeral for each expression.

1. $3 + 5 \cdot 2$ **2.** $5 \cdot 6 - 2$ **3.** $(5 + 3) \cdot 8$

4. $5 + 3(6 + 4)$ **5.** $6 + 3 \cdot 5 + 8 - 2 \cdot 3 + 6 - 4 - 1$

6. What numbers may result from $1 + 3n$ if the replacement set for n is $\{0, 1, 2, 3, 4, \cdots\}$?

Answers: **1.** 13 **2.** 28 **3.** 64 **4.** 35 **5.** 24 **6.** $\{1, 4, 7, 10, 13, \cdots\}$

Checkpoint

1. What is a variable? What is the connection between a variable and a replacement set?

2. What is an algebraic expression? When does an expression such as $2 + 5x$ name a number?

3. What agreement is made on the order of operations in an algebraic expression?

Exercises

A If t is a variable whose replacement set is $\{1, 3, 5, 7\}$, find the set of numbers named by each of the following.

1. $4t$ **2.** t^2 **3.** $t - 1$ **4.** $\frac{t+1}{2}$

If n is a variable whose replacement set is the set of whole numbers, W, list the set of numbers named by each of the following.

5. $2n$ **6.** $2n + 1$ **7.** $2n + 2$

8. $3n$ **9.** $n + \frac{1}{2}$ **10.** $2n + \frac{1}{2}$

11. $\frac{n}{2} + 7$ **12.** $5(n + 3)$ **13.** $\frac{3+n}{2}$

14. $3n(5 + n)$ **15.** $\frac{n}{2}$ **16.** $\frac{n}{3} + \frac{2}{3}n$

17. If the replacement set is W, which of the following names all the even numbers? the odd numbers? the multiples of 3? none of these?

a. $3n$ **b.** $2n$ **c.** $2n + 1$ **d.** $3n + 1$

If the replacement set for the variable t is $\{0, 1, 2, 5\}$, what is the least number represented by each of the following?

18. t **19.** $t + t + t$ **20.** $t + 2$

21. $2t$ **22.** $t - t$ **23.** $t \cdot t$

B For which variables is a finite set of whole numbers the replacement set?

24. n represents the number of pounds of candy you buy at \$.89 a pound. } \$.89n is the total cost.

25. l represents the number of inches in the length of a rectangle whose width is 5 inches. } $5l$ is the number of square inches in the area of the rectangular region.

26. x represents the number of people in your class. } $10x$ is the total number of fingers on all the people in the class.

If $p = 3t$ is true and the replacement set for t is $\{1, 2, 3\}$, what is the greatest number represented by each of the following?

27. $p + 5$ **28.** $p + t$ **29.** $4 + t$

30. $5t$ **31.** $4t - p$ **32.** $7p - 4t$

33. $p - 3$ **34.** $p - p$ **35.** $4(p + t)$

List the replacement set necessary so that each of the following names a whole number.

36. $a + 2$ **37.** $x \div 2$ **38.** $x \div x$

39. $0 \div a$ **40.** $\frac{b}{2} + \frac{1}{2}$ **41.** $(3x + 9) \div 2$

1–5 Sentences

You know that "I am an algebra student" is a true English sentence about you and algebra. The following are true *mathematical* sentences with a brief description of each.

a. $5 + 2 = 7$ A sentence using $=$, *is equal to*, is an **equality,** or an **equation.**

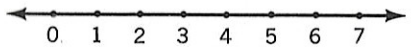
b. $8 > 5$ A sentence using $>$, *is greater than*, is an **inequality.**

c. $15 + 6 < 25$ A sentence using $<$, *is less than*, is an **inequality.**

d. $15 \neq 4$ A sentence using $\neq$, *is not equal to*, is an **inequality.**

The symbol $=$ means "is identical to" or "names the same thing as." The sentence $5 + 3 = 10$ is false because $5 + 3$ and 10 do not name the same number. The sentence can be made true by changing the symbol to $\neq$: $5 + 3 \neq 10$. (Note that a line through any symbol means "not.")

The relations *is greater than* and *is less than* can be visualized on the **number line.** To make a number line, match 0 with a point of a horizontal line, and choose a point to the right to match 1; then mark off congruent (equal in measure) segments to locate points for successive whole numbers. The number is called the **coordinate** of the corresponding point. The arrows at the ends indicate that the numbers go on indefinitely in both directions.



To compare numbers using the number line, note the following.

A given number is less than any number to its right.

A given number is greater than any number to its left.

Pick a number and place your pencil at the corresponding point. Name numbers that are greater than the one you chose; name some that are less than the one you chose.

The idea of using the number line to compare two numbers can be stated as an equation. For example, you can say that $8 > 5$ is true because there is a counting number, 3, that added to 5 makes 8; that is, $8 > 5$ is true because $8 = 5 + 3$ is true. The definition is as follows.

Definition If a and b are whole numbers, then $a > b$ means that there is a natural number c such that $b + c = a$ is true. If $a > b$ is true, then $b < a$ is true.

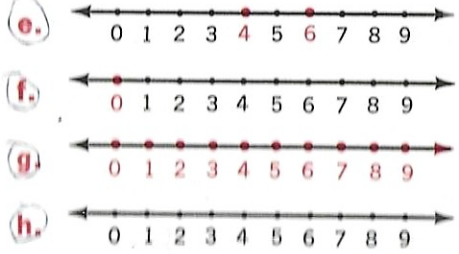
Look at sentences **e–h** below. You cannot tell whether they are true or false until elements of the replacement set are substituted for the variable. They are **open sentences.**

Definition The solution set of an open sentence is the set of numbers that make the sentence true.

Substitute elements of the replacement set, U, to explain how the solution set was obtained.

Sentence	Replacement set	Solution set
e. $x + 2 > 4$	$U = \{0, 2, 4, 6\}$	$\{4, 6\}$
f. $x + x = 0$	$U = \{0, 1, 2, 3, 4, \cdots\}$	$\{0\}$
g. $x + 3 > x$	$U = \{0, 1, 2, 3, 4, \cdots\}$	$\{0, 1, 2, 3, 4, \cdots\}$
h. $4 \div 3 = x$	$U = \{0, 1, 2, 3, 4, \cdots\}$	$\emptyset$

The graphs of sentences **e–h** are shown below.



The **graph of an open sentence** is the set of points on the number line whose coordinates make the sentence true. The graph of e is

two points, and of **f**, one point. The graph of **g** is all the points on the number line (the arrow shows that the graph continues indefinitely). The graph of **h** is no points because the solution set is empty.

Checkpoint

1. How is $=$ defined? $<$? $>$?
2. When is a sentence an equality? an inequality?
3. What is the coordinate of a point on a line?
4. What is the graph of an open sentence?

Exercises

A — Write *True* or *False* to describe each sentence.

1. $3 + 4 = 4 + 3$
2. $4 \cdot 3 = 3 \cdot 4$
3. $3 + (3 + 10) = (3 + 5) + 10$
4. $(4 + \frac{1}{5}) + \frac{4}{5} > 4 + (\frac{1}{5} + \frac{4}{5})$
5. $(2 \cdot 3) \cdot 5 < 2 \cdot (3 \cdot 6)$
6. $18 + 96 = 95 + 18$
7. $632 + 49 = 49 + 632$
8. $0 \cdot 632 > 632$
9. $0 + 632 = 632$
10. $1 \cdot 3\frac{1}{2} = 3\frac{1}{2}$
11. $12 + 44 < 43 + 12$
12. $27 \cdot 9 > 9 \cdot 26$

— If the replacement set is $U = \{2, 4, 6, 8, 10\}$, list the elements in the solution set of each of the following sentences.

13. $x + 2 < 10$
14. $n \neq 3 + 5$
15. $a + 2 = 2 + a$
16. $3x > 15$
17. $w + w = 8$
18. $t = 8$
19. $x \neq 2 + x$
20. $w + 5 = 7$

21. Which of the above sentences are equations? Which sentences are inequalities?
22. Look at the following. The replacement set is U.

$$U = \{1, 3, 5, 7\}$$

$$3x + 1 > 16$$

$3(1) + 1 > 16$ $\quad$ $3(3) + 1 > 16$
$3(5) + 1 > 16$ $\quad$ $3(7) + 1 > 16$

   a. Which of the above inequalities is an open sentence?
   b. Identify the replacement set.
   c. Which sentences are true? Which are false?
   d. List the elements of the solution set of the open sentence.

23. Write *True* or *False* to describe the sentence that results when x is replaced by 3.

   a. $3x = 9$ $\quad$ b. $x + 9 = 11$ $\quad$ c. $x - 1 = 2$
   d. $\frac{x}{2} = 1\frac{1}{2}$ $\quad$ e. $3x - x = 7$

— If the replacement set is $U = \{0, 1, 2, 3, 4, 5, \cdots\}$, write the solution set by the roster method.

24. $x + 3 > 15$
25. $x + 2 = 10$
26. $x + 2 > 10$
27. $10 + x = 30$
28. $10 + x < 30$
29. $x + 3 < x + 4$
30. $x + 1 \leq 5$
31. $3x \not< 9$
32. $x + 5 \neq 6$
33. $25 < 2x + 3x$

B — If x is a variable whose replacement set is $\{1, 2, 3, 4, 5, 6\}$, find the solution set for each of the open sentences below.

34. $\frac{x}{2} = 1$
35. $x^2 = 25$
36. $\frac{x}{10} < 1$
37. $x \not< 5 - 2$
38. $3x \not> \frac{x^2}{2}$

— Suppose that the replacement set for x and y is the set of whole numbers. Tell whether each of the following sentences is *True* or *False*. If the sentence is false, give an example that proves it is false. The symbol $\forall x\ \forall y$ means that for every pair of members chosen for x and for y, the sentence is true.

39. $\forall x\ \forall y$ $\quad x + y = y + x$
40. $\forall x\ \forall y$ $\quad (x + y) + x = x + (y + x)$
41. $\forall x\ \forall y$ $\quad x + y$ is one and only one whole number.
42. $\forall x\ \forall y$ $\quad xy$ is one and only one whole number.
43. $\forall x\ \forall y$ $\quad x + y > x + x$
44. $\forall x\ \forall y$ $\quad x \cdot y > x \cdot x$
45. $\forall x$ $\quad 1 \cdot x = x$
46. $\forall y$ $\quad 0 + y = y$
47. $\forall x$ $\quad$ There is a whole number a such that $xa = 0$.
48. $\forall x$ $\quad$ There is a whole number a such that $x + a = x$.
49. $\forall x$ $\quad$ There is a whole number a such that $xa = 1$.

1–6 The Set of Fractional Numbers

Tell why the set of whole numbers is not sufficient to answer these questions.

a. What is the measure of $\overline{AB}$ when $\overline{AC}$ measures one unit?

b. What part of the square region is shaded?

c. What number is the quotient $4 \div 5$?

Fractional numbers can be used for the answers.

For **a**, does the natural number 3 name the number of parts of equal length in $\overline{AC}$, the entire segment? Does the whole number 2 name the number of parts of equal length in $\overline{AB}$, the segment being considered? Then what does the **ordered pair** (2, 3) name? Is the fractional number $\frac{2}{3}$ the measure of $\overline{AB}$?

For **b**, what does 4 name; what does 2 name? What does the ordered pair (2, 4) tell about the shaded region in **b**? Does the fractional number $\frac{2}{4}$ describe the shaded part of the region in relation to the whole region?

What do the ordered pair (4, 5) and $4 \div 5$ have in common? Is the fractional number $\frac{4}{5}$ the quotient of 4 divided by 5?

The pairs of numbers (2, 3), (2, 4), and (4, 5) are called *ordered pairs* because the numbers are given in a particular order: 2 out of 3 parts, 2 out of 4 parts, and so forth.

In **a**, you compared 2 to 3; in **b**, you compared 2 to 4; in **c**, you compared 4 to 5. Each of these comparisons is a **ratio** of a whole number to a natural number. The ratio is the quotient of two numbers. Why must the second number be a natural number? This leads to the formulation of a definition for fractional numbers.

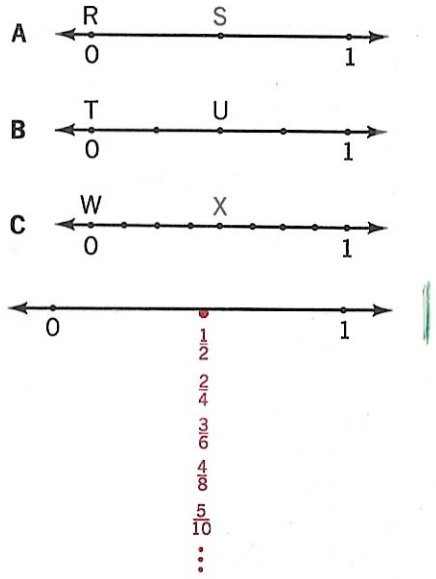
Definition A fractional number is a number that can be named as the ratio $\frac{x}{y}$, where $x \in W$ and $y \in N$.

As you know, when the fractional number is named by a ratio $\frac{x}{y}$, the name is a **fraction,** x being the **numerator** and y the **denominator.**

Can you name 10 as the ratio of 10 to 1, or as $\frac{10}{1}$, or as $\frac{20}{2}$? Can you name any whole number as a fraction?

The set of whole numbers is a subset of the set of fractional numbers.

What does the ordered pair (1, 2) tell about A at the right? What does (2, 4) tell about B? What does (4, 8) tell about C? Do the fractions $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ name the same part of the unit segment? Then the fractions $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ are names for one fractional number; they are **equivalent fractions.** The names $\frac{1}{2}$, $\frac{2}{4}$, $\frac{4}{8}$, and other names for the same fractional number are shown matched with one point on the number line. Does the number have five other names? a thousand other names? a million other names?



Another way of naming fractional numbers that are named as tenths, hundredths, thousandths, and so forth, is by decimals. As you recall, the places after the decimal point are called tenths, hundredths, and so forth. Thus, $\frac{3}{10} = 0.3$, $\frac{46}{100} = 0.46$, $\frac{79}{1000} = 0.079$, and $\frac{129}{100} = 1.29$ are true. Notice that 0.2, 0.20, 0.200, and so forth, all name the same fractional number. Why?

The set of names for any fractional number is an infinite set. Each fraction names an ordered pair — a whole number and a natural number.

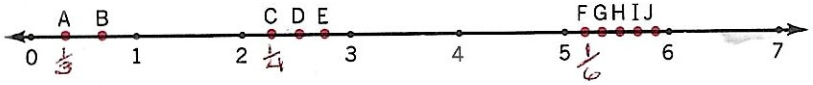
Checkpoint

1. What is a fractional number?
2. What is an ordered pair?
3. What is a fraction?
4. What are equivalent fractions?
5. How many equivalent fractions can there be in a set?

Exercises

A **1.** Give five names for the fractional number $\frac{7}{8}$.

2. What fractional number is the coordinate of each lettered point?



Name each fractional number shown below in three ways.

3. $\frac{2}{7}$ **4.** 62 **5.** 6.7 **6.** 0.000001

7. 3.82 **8.** 13.205 **9.** 0.623 **10.** $\frac{23}{5}$

Write a numeral for the variable to make each sentence true.

11. $\frac{1}{2} = \frac{n}{16}$ **12.** $0.14 = \frac{t}{100}$ **13.** $2.001 = \frac{y}{1000}$ **14.** $\frac{1}{4} = \frac{5}{x}$

Graph each of the following sets on a number line.

15. $\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$ **16.** $\{0, 0.5, 1, 1.5, 2\}$

17. Classify each number in Exercises 15 and 16 as a whole number, W, a natural number, N, or a fractional number, F. Some numbers may be classified in more than one way.

B **18.** Show a one-to-one correspondence between the set of natural numbers and the set of names for $\frac{1}{2}$.

$$\{\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \cdots\}$$

What is the number of elements in the given set of names?

1–7 Addition, Multiplication, and Equality

Probably you recall that fractional numbers showing the same denominator can be added by merely adding the numerators.

Definition When a and c are whole numbers and b is a counting number, then the following is true.

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

Thus, $\frac{7}{12} + \frac{4}{12}$ is equal to $\frac{11}{12}$. But can you add $\frac{7}{12}$ and $\frac{1}{3}$? You can if you recognize that $\frac{1}{3}$ is just another name for $\frac{4}{12}$.

It is not always easy to recognize names with common denominators for fractional numbers. Here multiplication can be useful. Recall that the product of two fractional numbers is the product of their numerators divided by the product of their denominators.

Definition If a and c are whole numbers and b and d are natural numbers, then the following is true.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

You may use the definition of multiplication when you rename a fractional number by multiplying it by a name for 1.

EXAMPLE 1. Do $\frac{35}{77}$ and $\frac{5}{11}$ name the same fractional number?

$$\frac{35}{77} \stackrel{?}{=} \frac{5}{11}$$
$$\stackrel{?}{=} \frac{5}{11} \cdot 1, \text{ or } \frac{5}{11} \cdot \frac{7}{7}$$
$$\frac{35}{77} = \frac{35}{77}$$

This shows that $\frac{35}{77}$ and $\frac{5}{11}$ do name the same fractional number.

Here is another way to check equality for fractional numbers.

Definition When $a, c \in W$ and $b, d \in N$, $\frac{a}{b} = \frac{c}{d}$ is true if and only if $ad = bc$ is true.

EXAMPLE 2. Do $\frac{2}{3}$ and $\frac{8}{12}$ name the same fractional number?

Examine the products $2 \cdot 12$ and $3 \cdot 8$. Is it true that $2 \cdot 12$ equals $3 \cdot 8$? Then $\frac{2}{3}$ and $\frac{8}{12}$ do name the same fractional number.

If you must add two fractional numbers, you may need to find equivalent fractions with a common denominator.

EXAMPLE 3. Add $\frac{1}{5}$ and $\frac{3}{8}$.

You must find equivalent fractions for each number. Note that 40 is a multiple of each denominator. Therefore, each given number can be multiplied by a name for 1 to have a denominator of 40, and the problem may be rewritten.

$$\frac{1}{5} + \frac{3}{8} = \frac{1}{5} \cdot \frac{8}{8} + \frac{3}{8} \cdot \frac{5}{5}$$
$$= \frac{8}{40} + \frac{15}{40} = \frac{23}{40}$$

You know that $\frac{1}{2}$ is greater than $\frac{1}{3}$. How can you tell which of *any* two fractional numbers is greater; that is, how can you order them?

EXAMPLE 4. Which is greater, $\frac{5}{11}$ or $\frac{4}{10}$?

Examine the products $5 \cdot 10$ and $11 \cdot 4$. Is $5 \cdot 10 > 11 \cdot 4$ true? If so, then $\frac{5}{11} > \frac{4}{10}$ is true.

Definition When a, $c \in W$ and b, $d \in N$, $\frac{a}{b} > \frac{c}{d}$ is true if and only if $ad > bc$ is true, and $\frac{a}{b} < \frac{c}{d}$ is true if and only if $ad < bc$ is true.

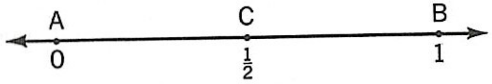
Use the definition to show that these sentences are true.

a. $\frac{4}{5} > \frac{1}{5}$ b. $\frac{3}{11} > \frac{2}{10}$ c. $\frac{9}{10} < \frac{8}{8}$

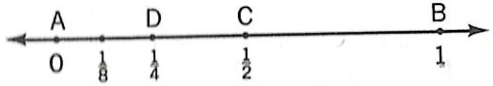
Between some pairs of whole numbers, it is impossible to find another number of the same set. This means that the set of whole numbers is **discrete.** This is not true of the fractional numbers. The fractional numbers are not discrete.

Between any two fractional numbers, another fractional number can be found. Between $\frac{1}{2}$ and $\frac{7}{8}$, the fractional number $\frac{11}{16}$ can be inserted. Between $\frac{11}{16}$ and $\frac{7}{8}$, you can insert $\frac{3}{4}$. This property of fractional numbers is the **density property.** A set is *dense* if there is always another member of the set between any two members.

The density property can be illustrated by using the number line. For example, between A and B, there is another point, C, that is halfway between A and B. If A matches 0 and B matches 1, then C matches $\frac{1}{2}$.



If you look at points A and C, you will see there is a point halfway between them, call it D, and it matches $\frac{1}{4}$; there is a point halfway between A and D that matches $\frac{1}{8}$; and so on.



This pattern continues indefinitely for any pair of different fractional numbers.

Exercises

A

1. Which is true?

a. $\frac{12}{17} > \frac{7}{9}$ b. $\frac{10}{17} < \frac{7}{9}$ c. $\frac{13}{17} = \frac{7}{9}$

2. Insert one fractional number between $\frac{5}{9}$ and $\frac{6}{8}$.

Add or multiply as directed.

3. $\frac{3}{5} + \frac{4}{5}$ 4. $\frac{3}{4} \times \frac{3}{7}$ 5. $\frac{1}{6} + \frac{1}{2}$

6. $\frac{7}{15} + \frac{2}{15}$ 7. $\frac{3}{8} \times \frac{3}{8}$ 8. $\frac{3}{8} + \frac{3}{15}$

9. $\frac{2}{3} + \frac{3}{4}$ 10. $\frac{1}{6} \times \frac{1}{4}$ 11. $\frac{1}{6} + \frac{1}{4}$

12. $\frac{3}{20} + \frac{2}{10}$ 13. $\frac{5}{10} + \frac{10}{100}$ 14. $0.3 + 0.4$

15. $0.25 + 0.5$ 16. 0.25×2.4 17. 1.02×2.37

The **mixed form** of a fractional number is one in which the number is named as the sum of a whole number and a fraction, without writing the addition sign. For example, $2\frac{1}{4}$ is a mixed form for the number also named by $\frac{9}{4}$ or by 2.25. Rename each of the following as mixed forms.

18. $\frac{5}{3}$ 19. $\frac{72}{5}$ 20. 2.5 21. 7.309

Replace each ● by one of the symbols $<$, $>$, or $=$ to make each sentence below true. You should be able to do most of these without computation.

22. $\frac{2}{3}$ ● $\frac{4}{6}$ 23. $\frac{111}{50}$ ● $\frac{112}{50}$

24. $\frac{5}{7}$ ● $\frac{5}{8}$ 25. $8 + 2\frac{1}{2}$ ● $12 - 1\frac{1}{2}$

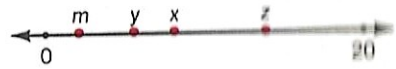
26. $6\frac{3}{4} - 6\frac{3}{4}$ ● 0 27. $15\frac{1}{2}$ ● 15.4998

28. $3 \times 4\frac{1}{2}$ ● $4\frac{1}{2} \times 3$ 29. 2.0001 ● 2.00001

30. $6 \div 7$ ● $\frac{6}{7}$ 31. $\frac{1}{2} \div 7$ ● $\frac{1}{2} \div 5$

32. $\frac{2}{3} \div 4$ ● $\frac{2}{3} \times \frac{1}{4}$ 33. $\frac{5}{6} \div \frac{2}{5}$ ● $\frac{5}{6} \times \frac{2}{5}$

The variables m, x, y, and z represent fractional numbers between 0 and 20. The numbers are arranged as shown at the right. Replace the ● with $<$, $>$, or $=$ to make true sentences.



34. m ● 20 35. 0 ● x 36. x ● m 37. m ● y

38. m ● z 39. z ● y 40. m ● x 41. 20 ● y

42. x ● x 43. $3m$ ● 0 44. $5y$ ● 0 45. $\frac{1}{4}m$ ● 20

— What set of numbers is represented when the variable is replaced in turn by each element of the replacement set $\{\frac{1}{2}, 2\frac{1}{2}, 8\}$?

46. $2n$ **47.** $3n + 1$ **48.** n^2 **49.** $3n^2$

50. $\frac{n}{2}$ **51.** $\frac{2}{n}$ **52.** $\frac{n+2}{4}$ **53.** $\frac{4}{n+1}$

— If the replacement set is $\{0.2, 0.3, 0.4\}$, what set of numbers is named by each expression in column **a**? in column **b**?

a	b
54. $4n + 3n$	$7n$
55. $n^2 + n^2$	$2n^2$
56. $(4n + 3n) + 2n^2$	$7n + 2n^2$

57. The division example shows how the decimal name for $\frac{1}{7}$ is found. What digit replaces the question mark? The numerals in color show the remainders. What will be the next six remainders? Then, what will be the next six digits in the quotient?

$$\begin{array}{r} 0.142857? \\ 7\overline{)1.0000000\cdots} \\ 7 \\ \hline 30 \\ 28 \\ \hline 20 \\ 14 \\ \hline 60 \\ 56 \\ \hline 40 \\ 35 \\ \hline 50 \\ 49 \\ \hline 10 \end{array}$$

Is $\frac{1}{7} = 0.142857\overline{142857}$ true if the bar means that the block of digits beneath the bar is repeated over and over? The decimal name for a number such as $\frac{1}{7}$ is a **repeating decimal.**

The decimal name for $\frac{1}{8}$ is 0.125. This is a **terminating decimal.**

— Name each fractional number below with a decimal. Use a bar to indicate a block of digits that repeats.

58. $\frac{3}{7}$ **59.** $\frac{9}{25}$ **60.** $\frac{1}{3}$ **61.** $\frac{1}{9}$ **62.** $\frac{2}{11}$ **63.** $\frac{11}{20}$

B — The example at the right shows how the repeating decimal $x = 0.12\overline{12}$ can be expressed as a fractional number. Express each of the following in the same way.

64. $0.3\overline{3}$ **65.** $0.32\overline{32}$

66. $635.15\overline{15}$ **67.** $0.83\overline{3}$

68. $0.123\overline{123}$ (*Hint:* Multiply by 1000 instead of 100. Why?)

69. $6.872\overline{872}$

$$\begin{aligned} x &= 0.12\overline{12} \\ 100x &= 100(0.12\overline{12}) \\ &= 12.12\overline{12} \end{aligned}$$

Subtract.

$$\begin{aligned} 100x &= 12.12\overline{12} \\ x &= 0.12\overline{12} \\ \hline 99x &= 12 \\ x &= \tfrac{12}{99}, \text{ or } \tfrac{4}{33} \end{aligned}$$

1–8 Postulates for Addition and Multiplication

From your previous school work, you have probably learned the properties of the operations on the fractional numbers. The properties provide reasons for much of your work in algebra. These properties are assumptions that are made about the operations of addition and multiplication. An assumption is called a **postulate.** The postulates are named and summarized in the table.

Read each postulate and its name. The symbol $\forall x$ is read "for every x" and the symbol $\forall x\ \forall y$ is read "for every x and every y."

Postulates for Addition and Multiplication of Fractional Numbers

Postulate	Name
1–1 $\forall x \in F\ \ \forall y \in F\ \ x + y$ **is a fractional number and only one fractional number.**	Closure postulate for addition.
1–2 $\forall x \in F\ \ \forall y \in F\ \ xy$ **is a fractional number and only one fractional number.**	Closure postulate for multiplication.
1–3 $\forall x \in F\ \ \forall y \in F$ $x + y = y + x$	Commutative postulate for addition.
1–4 $\forall x \in F\ \ \forall y \in F\ \ xy = yx$	Commutative postulate for multiplication.
1–5 $\forall x \in F\ \ \forall y \in F\ \ \forall z \in F$ $x + (y + z) = (x + y) + z$	Associative postulate for addition.
1–6 $\forall x \in F\ \ \forall y \in F\ \ \forall z \in F$ $x(yz) = (xy)z$	Associative postulate for multiplication.
1–7 $\forall x \in F\ \ \forall y \in F\ \ \forall z \in F$ $x \cdot (y + z) = (xy) + (xz)$	Distributive postulate for multiplication over addition.
1–8 $\forall x \in F$ $x + 0 = 0 + x = x$	Additive identity postulate (postulate for 0 in addition).
1–9 $\forall x \in F\ \ 1 \cdot x = x \cdot 1 = x$	Multiplicative identity postulate (postulate for 1 in multiplication).
1–10 $\forall x \in F\ \ x \neq 0$, **there exists** y **in F such that** $xy = 1$.	Multiplicative inverse postulate.

Note that if two fractional numbers are multiplicative inverses, their product is the identity element for multiplication, 1. The

$\frac{1}{x} = \frac{1}{2}$ $\qquad$ $\frac{1}{x} = \frac{1}{\frac{3}{4}}$, or $\frac{4}{3}$

$2 \cdot \frac{1}{2} = 1$ $\qquad$ $\frac{3}{4} \cdot \frac{4}{3} = 1$

Another name for the multiplicative inverse is **reciprocal.**

You should be able to explain each postulate in your own words and give an example like those below.

	Postulate	Example
1–1	Closure postulate for addition.	$(\frac{1}{4} + \frac{1}{8})$, or $\frac{3}{8}$, is a fractional number and only one fractional number.
1–2	Closure postulate for multiplication.	$(4 \cdot \frac{1}{5})$, or $\frac{4}{5}$, is one and only one fractional number.
1–3	Commutative postulate for addition.	$19 + 0.5 = 0.5 + 19$
1–4	Commutative postulate for multiplication.	$0.5 \cdot 0.6 = 0.6 \cdot 0.5$
1–5	Associative postulate for addition.	$(\frac{1}{3} + \frac{1}{5}) + \frac{1}{2} = \frac{1}{3} + (\frac{1}{5} + \frac{1}{2})$
1–6	Associative postulate for multiplication.	$(\frac{2}{5} \cdot \frac{7}{9}) \cdot \frac{5}{3} = \frac{2}{5} \cdot (\frac{7}{9} \cdot \frac{5}{3})$
1–7	Distributive postulate for multiplication over addition.	$3(20 + 4) = (3 \cdot 20) + (3 \cdot 4)$
1–8	Additive identity postulate.	$\frac{196}{9} + 0 = 0 + \frac{196}{9} = \frac{196}{9}$
1–9	Multiplicative identity postulate.	$1 \cdot 15\frac{1}{2} = 15\frac{1}{2} \cdot 1 = 15\frac{1}{2}$
1–10	Multiplicative inverse postulate.	$6 \cdot \frac{1}{6} = \frac{1}{6} \cdot 6 = 1$

Checkpoint

1. Which postulate deals with the order of addends? of factors?
2. Do the associative postulates state anything about the order of the addends? Why are these postulates sometimes called the "grouping" postulates?
3. Which one postulate does not hold for whole numbers?
4. Under what conditions is a set of numbers closed?
5. What is meant by *identity element*?

A — In each of the following, write the numeral for the number that makes the sentence true. Then name the postulate that the completed sentence illustrates.

1. $12 \cdot \frac{4}{5} = \frac{4}{5} \cdot n$
2. $6(2 + x) = (6 \cdot 2) + (6 \cdot 1)$
3. $5 + 10 = a + 5$
4. $\frac{1}{6} \cdot (\frac{1}{5} \cdot \frac{1}{4}) = (c \cdot \frac{1}{5}) \cdot \frac{1}{4}$
5. $4 + 6 = 6 + b$
6. $\frac{7}{8} \cdot (\frac{9}{2} \cdot \frac{8}{5}) = (\frac{7}{8} \cdot \frac{8}{2}) \cdot m$
7. $\frac{2}{3} \cdot h = 1$
8. $9 + (3 + 1) = (x + 3) + 1$

— Write the name of the postulate that is a reason why each sentence is true.

9. $6 + 8 = 8 + 6$
10. $(10 \cdot 5) \cdot 4 = 10 \cdot (5 \cdot 4)$
11. $3(10) = (10)3$
12. $8 \cdot (2 + \frac{1}{2}) = (8 \cdot 2) + (8 \cdot \frac{1}{2})$
13. $(6 \cdot 0)5 = 6(0 \cdot 5)$
14. $(1 + 3) + (6 \cdot 7) = (3 + 1) + (6 \cdot 7)$
15. $\frac{2}{3} \cdot 1\frac{1}{2} = 1$
16. $(6 \cdot 3) + (8 \cdot 2) = (8 \cdot 2) + (6 \cdot 3)$
17. $(6 + 0) + \frac{2}{3} = 6 + \frac{2}{3}$
18. $(6 + \frac{1}{3}) + (\frac{4}{3} \cdot 2) = (\frac{1}{3} + 6) + (\frac{4}{3} \cdot 2)$
19. $6 \cdot 1 = 6$
20. $8(\frac{1}{2} + 3) = (8 \cdot \frac{1}{2}) + (8 \cdot 3)$
21. $0 + 0 = 0$
22. $(3 \cdot 4) + (3 \cdot 5) = 3 \cdot (4 + 5)$
23. $8 + 100 = 100 + 8$
24. $\frac{3}{2} \cdot 9 + \frac{3}{2} \cdot 30 = \frac{3}{2}(9 + 30)$

B — A combination of postulates is needed as reasons why each of the following is true. Write the names of the postulates.

25. $8(5 \cdot 6) = 6(8 \cdot 5)$
26. $(3 \cdot 2) + (3 \cdot 7) = 3(7 + 2)$
27. $\frac{9}{2} + \frac{5}{2} = \frac{1}{2}(5 + 9)$
28. $6(2 + 0) = 0 + (6 \cdot 2)$
29. $8(1 \cdot \frac{4}{5}) = \frac{32}{5}$
30. $6 + (0.75 + \frac{1}{3}) = \frac{1}{3} + (6 + 0.75)$
31. $\frac{2}{7}(3\frac{1}{2}) + 10 = 10 + 1$
32. $2 \cdot 3 \cdot (1 + 4) = [(2 \cdot 1) + (2 \cdot 4)]3$

C — In the following exercises, there is hidden one true sentence showing that the distributive postulate holds for the given operations. Can you find this true sentence?

Example. Addition is distributive with respect to subtraction.

To express this sentence algebraically, you replace $\cdot$ by $+$ and $+$ by $-$ in the pattern $a \cdot (b + c) = ab + ac$. Thus, an example expressing the meaning of the above sentence might be $6 + (3 - 2) = (6 + 3) - (6 + 2)$. It is apparent that this sentence is not true. Therefore, you conclude that addition is not distributive with respect to subtraction.

33. Division is distributive with respect to division.
34. Division is distributive with respect to multiplication.

35. Division is distributive with respect to subtraction.
36. Subtraction is distributive with respect to addition.
37. Addition is distributive with respect to addition.
38. Subtraction is distributive with respect to subtraction.
39. Multiplication is distributive with respect to subtraction.

1–9 Using the Postulates

There are three more postulates, the Postulates of Equality, for which you will find frequent use.

	Postulate	Name
1–11	$\forall a \in F \quad a = a$	Reflexive postulate.
1–12	$\forall a \in F \quad \forall b \in F$ **If** $a = b$, **then** $b = a$.	Symmetric postulate.
1–13	$\forall a \in F \quad \forall b \in F \quad \forall c \in F$ **If** $a = b$ **and** $b = c$, **then** $a = c$.	Transitive postulate.

You can use these postulates with the postulates for addition and multiplication to simplify written computation or even to allow computation to be done mentally. For instance, the associative postulate makes it easier to find the following sum.

$$\begin{aligned}(13 + 97) + 3 &= 13 + (97 + 3)\\ &= 13 + \quad 100\\ &= 113\end{aligned}$$

You may be familiar with using the distributive postulate to simplify multiplication.

$$\begin{aligned}21 \times 13 &= 21(10 + 3)\\ &= (21 \times 10) + (21 \times 3)\\ &= \quad 210 \quad + \quad 63\\ &= 273\end{aligned}$$

Sometimes, it is better to reverse the steps and simplify in the following way, since by the symmetric postulate, the distributive postulate can be reversed.

$$\begin{aligned}(10 \times 14) + (10 \times 4) &= 10(14 + 4)\\ &= 10 \times 18\\ &= 180\end{aligned}$$

The distributive postulate, $x \cdot (y + z) = xy + xz$, is often called the **left distributive postulate.** Can you complete the following proof of the **right distributive property,** $(y + z) \cdot x = yx + zx$?

Statements	*Reasons*
1. $(y + z) \cdot x = x \cdot (y + z)$	1. Commutative postulate.
2. $x \cdot (y + z) = xy + xz$	2. ?
3. $(y + z) \cdot x = xy + xz$	3. Transitive postulate.
4. $(y + z) \cdot x = yx + zx$	4. ?

Why is $(y + z) \cdot x = yx + zx$ called the *right distributive property*? Many other uses of the postulates will be shown in the Exercises.

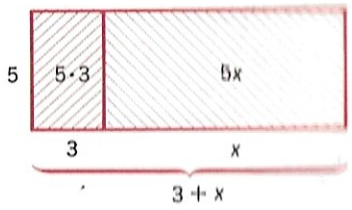
Exercises

A — Make use of the commutative and associative postulates of addition and multiplication to find the simplified name for each given expression. You should be able to write the answers with no other written work.

1. $(2 + 29) + 1$
2. $(18 + 6) + 14$
3. $(5365 + 1150) + 1850$
4. $(879 + 13) + 21$
5. $0.3(8 \cdot 10)$
6. $\frac{1}{5}(3 \cdot 100)$
7. $2(5 \cdot 19)$
8. $25(4 \cdot 12)$
9. $250(4 \cdot 12)$
10. $125(8 \cdot 65)$
11. $4(26 \cdot 25)$
12. $0.5(362 \cdot 2)$

13. A frame is $3 + x$ units long and 5 units wide. The area of the region enclosed by the frame is found by multiplying $3 + x$ by 5.

$$\begin{aligned}\text{Area} &= 5(3 + x)\\ &= 15 + \text{?}\end{aligned}$$



What is the other addend in the sum for the area? Find the area inside the frame when x is replaced by 7; by 19; by 265.

— Use the distributive postulate to name each number with a single numeral.

14. $3 \cdot (10 + 6)$
15. $3 \cdot (20 + 9)$
16. $2 \cdot (9 + \frac{1}{2})$
17. $8 \cdot (200 + 6)$
18. $20 \cdot (40 + 7)$
19. $\frac{3}{4} \cdot (3000 + 8)$

— Reverse the steps of simplification, using the distributive postulate.

20. $8(3) + 8(4)$ **21.** $10(9) + 10(0.2)$

22. $67(8) + 67(2)$ **23.** $86(97) + 86(3)$

— Use the distributive postulate to perform these computations. Do as much work mentally as you can.

Example. $8(32) = \underline{?}$

$$8(32) = 8(30 + 2) = 240 + 16 = 256$$

24. $3(21)$ **25.** $\frac{1}{4}(102)$ **26.** $11(205)$

27. $9(1009)$ **28.** $5(86)$ **29.** $9(1200)$

30. $5(52)$ **31.** $6(4\frac{2}{3})$ **32.** $12(72)$

— Use the right distributive property to name each algebraic expression below as an expression with no addition signs.

Examples. $5x + 8x = (5 + 8) \cdot x$
$= 13x$

$15y^2 + 8y^2 = (15 + 8) \cdot y^2$
$= 23y^2$

33. $3a + 7a$ **34.** $15y + 18y$

35. $14(xy) + 2(xy)$ **36.** $17m + 18m$

37. $635a + 22a$ **38.** $5 \cdot (x \cdot x) + 6 \cdot (x \cdot x)$

39. $15x^2 + 2x^2$ **40.** $13y^2 + 26y^2$

— Simplify both expressions in each item below.

41. $5 - (4 - 1)$ and $(5 - 4) - 1$

42. $30 - (10 - 6)$ and $(30 - 10) - 6$

43. $25 - (11 - 2)$ and $(25 - 11) - 2$

44. $2832 - (2762 - 13)$ and $(2832 - 2762) - 13$

45. According to your answers for Exercises 41–44, does the associative postulate hold for subtraction?

— Simplify both expressions in each item below.

46. $20 \div (4 \div 2)$ and $(20 \div 4) \div 2$

47. $144 \div (45 \div 9)$ and $(144 \div 45) \div 9$

48. $(35 \div 7) \div 5$ and $35 \div (7 \div 5)$

49. $10 \div (20 \div 4)$ and $(10 \div 20) \div 4$

50. According to your answers for Exercises 46–49, does the associative postulate hold for division?

51. Does the commutative postulate hold for subtraction? for division? Illustrate.

B — There is a transitive postulate that applies to "is greater than" and "is less than," in addition to the one for "is equal to." Use the following elements in the order given with $=$, $>$, or $<$ to make true sentences in "if . . . then" form.

Example. 1, 2, 3

If $1 < 2$ and $2 < 3$, then $1 < 3$.

52. $\frac{1}{2}, \frac{1}{3}, \frac{1}{9}$ **53.** $4 + 5, 9, 5 + 4$

54. 0.25, 0.2, 0.02 **55.** 1 ounce, 1 pound, 1 ton

56. 100, 10^2, $199 - 99$ **57.** 25 pennies, 5 nickels, 1 quarter

58. $\frac{2}{7}, \frac{3}{8}, \frac{4}{10}$ **59.** 6 inches, $\frac{1}{2}$ foot, $\frac{1}{6}$ yard

60. The formula for the area of a triangle is $A = \frac{1}{2}bh$, where A is the number of square units of area, b is the number of units in the base, and h is the number of units in the altitude. Have you ever made mistakes in finding $\frac{1}{2}bh$ by multiplying $\frac{1}{2}$ by b and $\frac{1}{2}$ by h? If you have a clear understanding of the associative postulate of multiplication, you will not make such errors.

$A = (\frac{1}{2}b)h$ OR $A = \frac{1}{2}(bh)$

Given $b = 10$ and $h = 6$.

$A = (\frac{1}{2} \cdot 10)6$
$= 5 \cdot 6$
$= 30$ square units

$A = \frac{1}{2}(10 \cdot 6)$
$= \frac{1}{2}(60)$
$= 30$ square units

When b is 10 and h is 5, $A = (\frac{1}{2}b)h$ is easier. But when b is 7 and h is 9, $A = \frac{1}{2}(bh)$ is easier. Explain.

Use the commutative postulate to show that $\frac{1}{2}(hb)$ and $(\frac{1}{2}h)b$ can also be used. When would it be sensible to use them?

61. A desk top is $x + y$ units long. Its width is r units. Its area may be written $r(\underline{?} + \underline{?})$. Use the distributive postulate and express the area of the desk top in a different way.

— Whether or not closure under an operation exists depends on both the set of elements in question and the operation used. Indicate which sets are closed under the specified operation. If the set is not closed, give a **counterexample** that shows this.

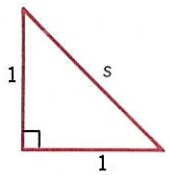
62. {0, 1, 2, 3}; addition.

63. {0, 1}; multiplication.

64. {0, 1}; addition.
65. {2}; subtraction.
66. {0, 1}; division.
67. {0}; division.
68. The set of all natural numbers that are multiples of 3; addition.
69. The set of all whole numbers that are multiples of 5; multiplication.
70. The set of natural numbers; subtraction.
71. The set of natural numbers; division.
72. The set of even whole numbers; multiplication.
73. The whole numbers; division.

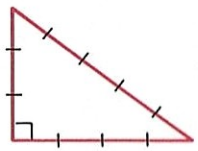
1–10 Pythagorean Theorem and Irrational Numbers

The triangle at the right is called a right triangle because it has one 90-degree angle. The side s, opposite the right angle, is called the **hypotenuse.** The other two sides are called **legs.** Each of the legs is the same length, one unit. What is the measure of the hypotenuse?



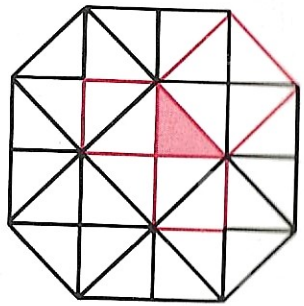
Make a number line, using the length of the legs of the right triangle as the unit. You can match the length of the hypotenuse with a point on the number line, but is there a fractional number that names this point?

Now look at another right triangle. The sides are marked off in units.

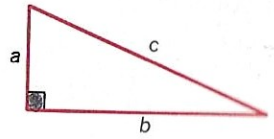


a. What is the length of the hypotenuse? What is the square of this length?
b. What is the length of one of the legs? What is the square of its length?
c. What is the length of the other leg? the square of its length?
d. Add the squares of the lengths of the legs together. How does this sum compare to the square of the hypotenuse? Is $9 + 16 = 5^2$ a true statement?

Look at the section of tile floor pictured here. All the triangles are right triangles. How does the sum of the areas of the squares on the two shorter segments of the shaded triangle compare with the area of the square on the longer side of the triangle? Is the sum of the two small squares outlined in color the same as the large square outlined in color?



The relationship between the lengths of the legs of a right triangle and the length of the hypotenuse is known as the Pythagorean theorem. Proving the theorem would take more algebra than you know now, but you can still use it. Refer to the figure at the right as you read Theorem 1–1.

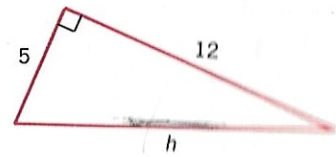


Theorem 1–1 (Pythagorean Theorem) If a, b, and c are the measures of the three sides of a right triangle, then $a^2 + b^2 = c^2$ is true, provided that c is the length of the hypotenuse.

Note how the theorem is used in the example below.

EXAMPLE. The sides of a right triangle are 5 units and 12 units. Find h, the length of the hypotenuse.

To find h, use the Pythagorean theorem $a^2 + b^2 = c^2$, replacing a by 5, b by 12, and c by h.



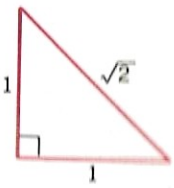
$$5^2 + 12^2 = h^2$$
$$25 + 144 = h^2$$
$$169 = h^2$$

Since $169 = h \cdot h$ is true, h is 13.

Check $5^2 + 12^2 = 13^2$
$25 + 144 = 169$

Now s, the measure of the hypotenuse in the triangle at the beginning of the section, can be found. Use the Pythagorean theorem, replacing a by 1, b by 1, and c by s.

$$1^2 + 1^2 = s^2$$
$$1 + 1 = s^2$$
$$2 = s^2$$
$$\sqrt{2} = s$$



The symbol $\sqrt{2}$ denotes the number s, such that $s \cdot s = 2$ is true, or $\sqrt{2} \cdot \sqrt{2} = 2$ is true. Likewise, $\sqrt{16}$ denotes the number x, such that $x \cdot x = 16$ is true. Since $4 \cdot 4$ equals 16, x is 4, or $\sqrt{16}$ is 4.

Recall that $\sqrt{x}$ denotes a number such that $\sqrt{x} \cdot \sqrt{x}$ is x. Is there a fractional number s for $\sqrt{2}$ such that $s \cdot s$ is 2? Try $1\frac{1}{2}$.

$$\tfrac{3}{2} \cdot \tfrac{3}{2} = \tfrac{9}{4}, \quad \text{or} \quad 2\tfrac{1}{4}$$

So $1\frac{1}{2}$ is too large. Try $1\frac{1}{4}$.

$$\tfrac{5}{4} \cdot \tfrac{5}{4} = \tfrac{25}{16}, \quad \text{or} \quad 1\tfrac{9}{16}$$

You can see that $1\frac{1}{4}$ is too small. You can try a few more examples, but you will find that there is no fractional number for $\sqrt{2}$.

Look at the members of the following set.

$$\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}, \sqrt{17}\}$$

None of these numbers can be expressed as a fractional number, yet each of them matches a point on the number line, as was shown for $\sqrt{2}$ at the beginning of the section. The members of the above set are members of the set of **irrational numbers.**

Definition An irrational number is one that cannot be expressed in the form of a ratio, $\frac{a}{b}$ $(a \in W,\ b \in N)$.

You cannot name an irrational number with a fractional number, or a terminating or a repeating decimal. You can, however, approximate irrational numbers by using decimals. (See the table on page 546.) A decimal approximation for $\sqrt{2}$ to the nearest thousandth, is $\sqrt{2} \approx 1.414$, where $\approx$ means "is approximately equal to." How does the computation at the right prove that $\sqrt{2} = 1.414$ is false?

$$\begin{array}{r} 1.414 \\ \times\ 1.414 \\ \hline 5656 \\ 1414 \\ 5656 \\ 1414 \\ \hline 1.999396 \end{array}$$

Now try these

— Which of the following numbers are fractional, F, and which are irrational, Ir? If the number is fractional, express it without the $\sqrt{\ }$.

1. $\sqrt{25}$ **2.** $\sqrt{24}$ **3.** $\sqrt{\frac{25}{9}}$ **4.** $\sqrt{81}$ **5.** $\sqrt{\frac{16}{36}}$ **6.** $\sqrt{11}$

7. If the two legs of a right triangle are 3 and 7, how long is the hypotenuse?

Answers: **1.** F; 5 **2.** Ir **3.** F; $\frac{5}{3}$ **4.** F; 9 **5.** F; $\frac{4}{6}$ **6.** Ir **7.** $\sqrt{58}$

Checkpoint

1. What does the Pythagorean theorem tell you about the relationship between the lengths of the sides of a right triangle?

2. What is an irrational number?

Exercises

A — Find the length c (a fractional number) of the hypotenuse of a right triangle, where a and b are the lengths of the other two sides.

	a	b	c
1.	6	8	?
2.	12	9	?
3.	12	5	?
4.	10	24	?
5.	15	20	?

— In each of the following exercises, c is an irrational number. Name the number for c by using a square root sign. Then use the table, and give a decimal approximation to the nearest thousandth.

	a	b	c
6.	1	1	? $\approx$?
7.	2	1	? $\approx$?
8.	2	3	? $\approx$?
9.	2	2	? $\approx$?
10.	3	3	? $\approx$?
11.	4	5	? $\approx$?
12.	10	11	? $\approx$?
13.	5	10	? $\approx$?
14.	2	4	? $\approx$?

B — Replace n with the smallest possible whole number to make a true statement.

15. $6 < \sqrt{39} < n$ **16.** $7 < \sqrt{63} < n$

17. $3 < \sqrt{12} < n$ **18.** $15 < \sqrt{229} < n$

— Replace x with the largest possible whole number to make a true statement.

19. $x < \sqrt{223} < 15$ **20.** $x < \sqrt{441} < 22$

21. $x < \sqrt{189} < 14$ **22.** $x < \sqrt{625} < 26$

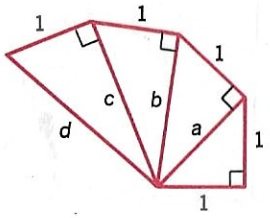
23. If the hypotenuse of a right triangle is 35 units long and one of its sides is 25 units long, how long is its other side? Leave the answer in radical form.

24. The hypotenuse of an isosceles right triangle (a right triangle with two equal sides) is 2 units. How long are its sides?

25. How high is a fence if the brace that supports it is 7 feet long and its ends are 3 feet from the foot of the fence and 3 feet from the top of the fence? Express the answer using $\sqrt{\ }$.

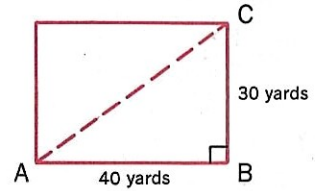
26. If the elements of $\{0, 1, 4, 9, 16, 25, 36, \cdots\}$ are called perfect squares, make a rule about which natural numbers have square roots that are irrational numbers.

C **27.** The figure shows a series of right triangles, each with an altitude of 1 unit. The hypotenuse of the first triangle becomes the base of the next triangle, and so on. Find the length of each hypotenuse, naming each irrational number with a square root sign.



28. The diagonal of a square is 2 units long. How long are its sides?

29. Look at the diagram at the right. If you walk at the speed of 8 feet per second, how much longer will it take you to walk from A to B and then from B to C, than directly from A to C?



CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

natural number (*2*)
square (*2*)
square root (*2*)
set (*2*)
element (*2*)
rule method (*3*)
roster method (*3*)
whole number (*3*)
empty set (*3*)
prime number (*5*)
composite number (*5*)
subset (*6*)
proper subset (*8*)
equal sets (*8*)
one-to-one correspondence (*12*)
equivalent sets (*12*)
finite set (*13*)
infinite set (*13*)
aleph-null (*14*)
variable (*16*)
replacement set (*16*)
algebraic expression (*17*)
term (*17*)
coefficient (*17*)
order of operation (*18*)
equation (*20*)
inequality (*20*)
number line (*20*)
coordinate (*20*)
open sentence (*21*)
solution set (*21*)
fractional number (*24*)
ordered pair (*24*)
ratio (*24*)
fraction (*24*)
equivalent fractions (*25*)
discrete (*28*)
dense (*28*)
mixed form (*29*)
repeating decimal (*30*)
terminating decimal (*30*)
postulate (*31*)
closure (*31*)
commutativity (*31*)
associativity (*31*)
distributivity (*31*)
identity element (*31*)
multiplicative inverse element, or reciprocal (*32*)
reflexive postulate (*34*)
symmetric postulate (*34*)
transitive postulate (*34*)
Pythagorean theorem (*39*)
irrational number (*40*)

2. Which of the following phrases describes a set: "the younger generation" or "teenagers"? Why?

3. Is the set of whole numbers finite or infinite? the set of even numbers? Is there a one-to-one correspondence between the whole numbers and the even numbers? Are the two sets equal? equivalent? Explain your answers.

4. What is the meaning of each of the following symbols?

a. $\in$ **b.** $\notin$ **c.** $\subset$ **d.** $\not\subset$ **e.** $\subseteq$ **f.** $\not\subseteq$ **g.** $\emptyset$

5. Write a rule to denote each of the following sets.

a. $G = \{3, 6, 9, 12, \cdots\}$
b. $H = \{2, 7, 12, 17, \cdots\}$
c. $J = \{4, 6, 8, 10, 12, 14\}$
d. $K = \{63, 70, 77, 84, 91, 98\}$

6. You are given the following sets.

W = {whole numbers}
P = {prime numbers}
O = {odd whole numbers}
N = {natural numbers}
E = {even whole numbers}
C = {composite numbers}

Replace ● by $\subseteq$ or $\not\subseteq$ to make each a true sentence.

a. N ● W
b. W ● N
c. $\{2, 4, 6, \cdots\}$ ● P
d. $\emptyset$ ● N
e. C ● E
f. $\{31, 37, 23, 29\}$ ● P

7. If the replacement set is X = {0, 1, 2, 3, 4, 5}, what set of numbers is named by each of the following expressions?

a. $7 + 3w$ **b.** $\frac{5w + 2}{2}$ **c.** $\frac{3w}{4} + \frac{5}{2w}$

8. If the replacement set is W = {0, 1, 2, · · ·}, give the solution set for each of the following and represent it on a number line.

a. $x + 5 = 17$ **b.** $x + x = 2x$ **c.** $x + 9 > 7$

d. $x + 7 < 5$ **e.** $3x + 2x = 24$ **f.** $x = x^2$

9. What is the postulate represented by each of the following?

a. $\forall a\ \forall b\quad a + b = b + a$ **b.** $\forall a\ \forall b\quad ab = ba$

c. $\forall a\quad 1 \cdot a = a \cdot 1 = a$ **d.** $\forall a\quad 0 + a = a + 0 = a$

e. $\forall a\ \forall b\ \forall c\quad (a + b) + c = a + (b + c)$

f. $\forall a\ \forall b\ \forall c\quad (ab)c = a(bc)$

10. If $\frac{a}{b}$ represents a fractional number, why may the replacement set for b not include 0?

11. What does it mean to say that the set of whole numbers is discrete and the set of fractional numbers is dense?

12. Replace ● with >, <, or = to make each sentence true.

a. $\frac{7}{9}$ ● $\frac{56}{72}$ **b.** $\frac{1}{2} + \frac{1}{5}$ ● $3\frac{1}{2} \times \frac{1}{5}$ **c.** $17\frac{2}{3}$ ● $\frac{52}{3}$

d. $\frac{5}{11}$ ● $\frac{25}{56}$ **e.** $\frac{9}{11} - \frac{1}{3}$ ● $\frac{4}{4} \cdot \frac{3}{4} + \frac{5}{16}$ **f.** 0.1251 ● $\frac{1}{8}$

Give a reason for each step in the following.

13. $3\frac{5}{8} + \frac{7}{8} = 3\frac{5}{8} + (\frac{3}{8} + \frac{4}{8}) = (3\frac{5}{8} + \frac{3}{8}) + \frac{4}{8} = 4 + \frac{1}{2} = 4\frac{1}{2}$

14. $7\frac{2}{3} \cdot 5 = 5 \cdot 7\frac{2}{3} = 5(7 + \frac{2}{3}) = (5 \cdot 7) + (5 \cdot \frac{2}{3}) = 35 + \frac{10}{3} = 38\frac{1}{3}$

15. $27 \cdot 8 = (20 + 7) \cdot 8 = 8 \cdot (20 + 7) = (8 \cdot 20) + (8 \cdot 7) = 216$

16. Which of the following are irrational numbers?

a. $\frac{17}{21}$ **b.** $.66\overline{6}$ **c.** $11\frac{1}{3}$ **d.** $\sqrt{81}$ **e.** $\sqrt{77}$

In each of the following, x and y are legs of a right triangle and h is the hypotenuse. Find the missing side.

17. $x = 13$ $y = 20$ $h = ?$

18. $x = 9$ $y = 12$ $h = ?$

19. $x = 36$ $y = ?$ $h = 60$

20. $x = ?$ $y = 12$ $h = 15$

CHAPTER TEST

Write the members of the following sets by the roster method.

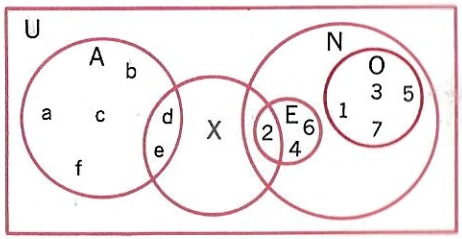
1. The set of whole numbers.

2. The set of perfect-square numbers.

3. The set of prime numbers greater than 5 and less than 25.

4. The set of even whole numbers greater than 4 and less than the square of 4.

Refer to the diagram at the right and write *True* or *False* to describe the following sentences.



5. $d \in A$ **6.** $X \subseteq A$

7. $E \nsubseteq A$ **8.** $4 \notin N$

9. $X \subset X$ **10.** $A \subseteq \emptyset$

11. $O \subset N$ **12.** $N \subseteq A$

13. $c \notin U$ **14.** X is equivalent to E.

Give the postulate or property that is the reason for each of the following statements.

	Statements	*Reasons*
15.	$2(3 + 1) + 6(\frac{1}{6}) = 6 + 2 + 6(\frac{1}{6})$	?
16.	$= 2 + 6 + 6(\frac{1}{6})$	?
17.	$= (2 + 6) + 1$	?
18.	$= 1 + (2 + 6)$	?

19. Write a single numeral for each of the following.

a. $\frac{2(3 + 4) - 13}{6}$ **b.** $2 + 6 \times 5 - 4 \div 2$

20. If the replacement set is {0, 1, 2, 3, 4, 5, 6}, give the solution set for each of the following.

a. $\frac{x^2}{3} < 8$ **b.** $x + 5 = 4$ **c.** $2x - 4 = \frac{x}{2} - 1$

21. Name each of the following fractions with a decimal. Use a bar to indicate a block of digits that repeats.

a. $\frac{1}{6}$ **b.** $\frac{2}{5}$ **c.** $\frac{1}{13}$

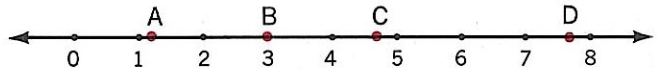
22. The lengths of two sides of a triangle are $2\frac{1}{2}$ and 6. Find the length of the hypotenuse.

23. Which of the following sets are closed under multiplication?

a. $\{0, 1, 2, 3, \cdots\}$ **b.** $\{0, 1\}$ **c.** $\{0, 1, \frac{1}{2}, \frac{1}{4}\}$

24. One leg and the hypotenuse of a right triangle measure $\sqrt{192}$ units and 16 units, respectively. How long is the other leg?

— Refer to the number line below and write $>$, $<$, or $=$ to make the following sentences true.



25. $A \bullet D$ **26.** $D \bullet 7$ **27.** $4 \bullet A$

28. $A \bullet B \bullet C$ **29.** $B \bullet 3$ **30.** $C \bullet 2$

31. Which pairs of fractions have a fractional number between them?

a. $\frac{3}{4}, \frac{1}{2}$ **b.** $\frac{15}{16}, \frac{7}{8}$ **c.** $\frac{32}{64}, \frac{16}{32}$

— Tell whether or not the following statements are true.

32. $\forall x \in W \quad x^2 - 2x = x(x - 2)$ **33.** $\forall y \in W \quad y + \frac{y}{2} = 2y$

ENRICHMENT

Finite Fields

For the set $X = \{a, b, c, \cdots\}$, the following postulates are true with respect to two operations, $*$ and $\circ$.

① $(a * b) \in X$ and $(a \circ b) \in X$	[Closure.]
② $(a * b) = (b * a)$ and $(a \circ b) = (b \circ a)$	[Commutativity.]
③ $(a * b) * c = a * (b * c)$ and $(a \circ b) \circ c = a \circ (b \circ c)$	[Associativity.]
④ $a \circ (b * c) = (a \circ b) * (a \circ c)$	[Distributivity.]
⑤ There exists an element z such that, for every $a \in X$, $a * z = a$, and there exists an element e $(e \neq z)$ such that, for every $a \in X$, $a \circ e = a$.	[Identity elements.]
⑥ For every $a \in X$, there exists an element i such that $a * i = z$, and for every $a \in X$ except z, there exists an element l such that $a \circ l = e$.	[Inverse elements.]

The set X that has these properties with respect to two operations $*$ and $\circ$ is a **field.** Examine the set $S = \{E, O\}$ with the two operations $+$ and $\times$ as defined in the following tables.

$+$	E	O
E	E	O
O	O	E

$\times$	E	O
E	E	E
O	E	O

You must check all six postulates to be sure the set is a field.

① Is S closed under addition and multiplication? Only E and O appear in the tables, so S is closed.

② Is S commutative under addition and multiplication?

$$E + O = O = O + E \qquad E \times O = E = O \times E$$

Hence, S is commutative.

You can also check for commutativity by the "diagonal-line test." If you compare the two halves of each chart, you see that they are exactly alike. (This is especially convenient with larger sets.)

③ To check associativity, you must check all the combinations for both operations, as shown below for addition.

$$E + (O + O) = (E + O) + O \qquad O + (E + E) = (O + E) + E$$
$$E + (O + E) = (E + O) + E \qquad O + (E + O) = (O + E) + O$$
$$E + (E + O) = (E + E) + O \qquad O + (O + E) = (O + O) + E$$
$$E + (E + E) = (E + E) + E \qquad O + (O + O) = (O + O) + O$$

Check $E + (O + O) = (E + O) + O$.

$$\begin{aligned} E + (O + O) &= (E + O) + O \\ E + \quad E \quad &= \quad O \quad + O \\ E &= E \end{aligned}$$

Check $E \times (O \times O) = (E \times O) \times O$.

$$\begin{aligned} E \times (O \times O) &= (E \times O) \times O \\ E \times \quad O \quad &= \quad E \quad \times O \\ E &= E \end{aligned}$$

Check the other combinations to be sure S is associative.

④ All possible combinations must also be checked to be sure S is distributive with respect to $+$ and $\times$. One is checked for you.

$$\begin{aligned} E \times (O + O) &= (E \times O) + (E \times O) \\ E \times \quad E \quad &= \quad E \quad + \quad E \\ E &= E \end{aligned}$$

⑤ Look at the tables to find the identity elements. The identity element for + is E.

$$O + E = O \qquad E + E = E$$

The identity element for $\times$ is O.

$$E \times O = E \qquad O \times O = O$$

⑥ Check for the inverse elements.

Is there an element that added to E gives E? Is there an element that added to O gives E?

Is there an element for every element of S except E (that leaves O) such that O times that element is O?

Thus, S under the operations defined in the tables is a field.

Exercises

1. Below are tables defining + and $\times$ for S = $\{i, j, k, l\}$.

+	i	j	k	l
i	k	l	i	j
j	l	k	j	i
k	i	j	k	l
l	j	i	l	k

$\times$	i	j	k	l
i	i	j	k	l
j	k	k	k	k
k	k	k	k	k
l	i	j	k	l

a. Is S closed under +? under $\times$?

b. Is S commutative under +? under $\times$?

c. Are the associative and distributive postulates satisfied?

d. Does each element have a multiplicative inverse? Explain.

e. Is k the additive identity element? Explain.

f. Can you find a product of two numbers, neither of which is the identity element, that is equal to the identity element? How does this differ from the whole numbers?

g. Is S with + and $\times$ as defined here a field? Why?

2. Below is an addition table and part of a multiplication table for a field of elements.

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

$\times$	a	b	c	d
a	a	a	a	a
b	a	?	c	d
c	a	?	?	b
d	a	d	b	?

Using the field postulates, fill in the rest of the multiplication table. (*Hint:* Since distributivity holds, the equation $(c \times c) + (c \times d) = c \times (c + d)$ is valid. The product $c \times c$ is unknown, but you can determine its value by evaluating the rest of the equation. Use a similar equation to determine $d \times d$.)

3. Is T = $\{0, 1, 2\}$ a field under addition and multiplication as defined below?

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\times$	0	1	2
0	2	0	1
1	0	1	2
2	1	2	0

4. Is T = $\{0, 1, 2\}$ a field with addition and multiplication as defined below?

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

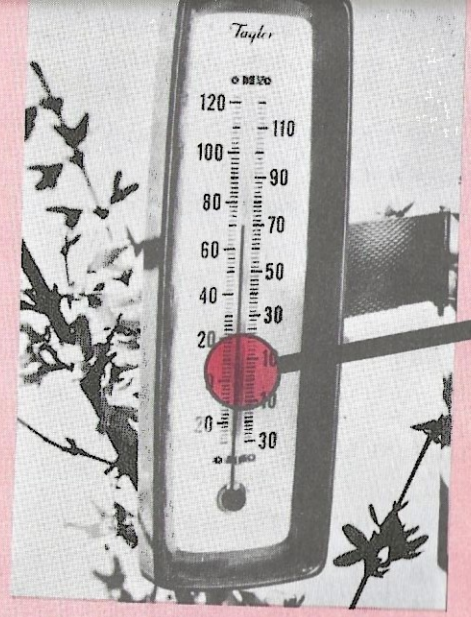
$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

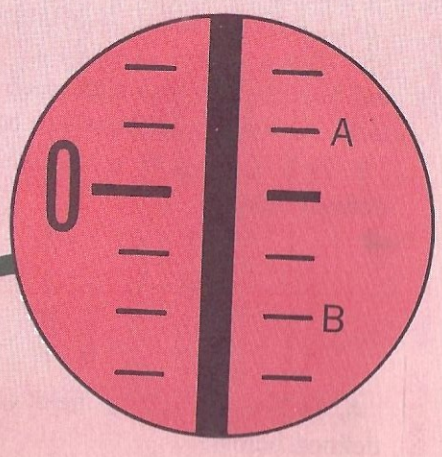
THE GROWTH OF MATHEMATICS

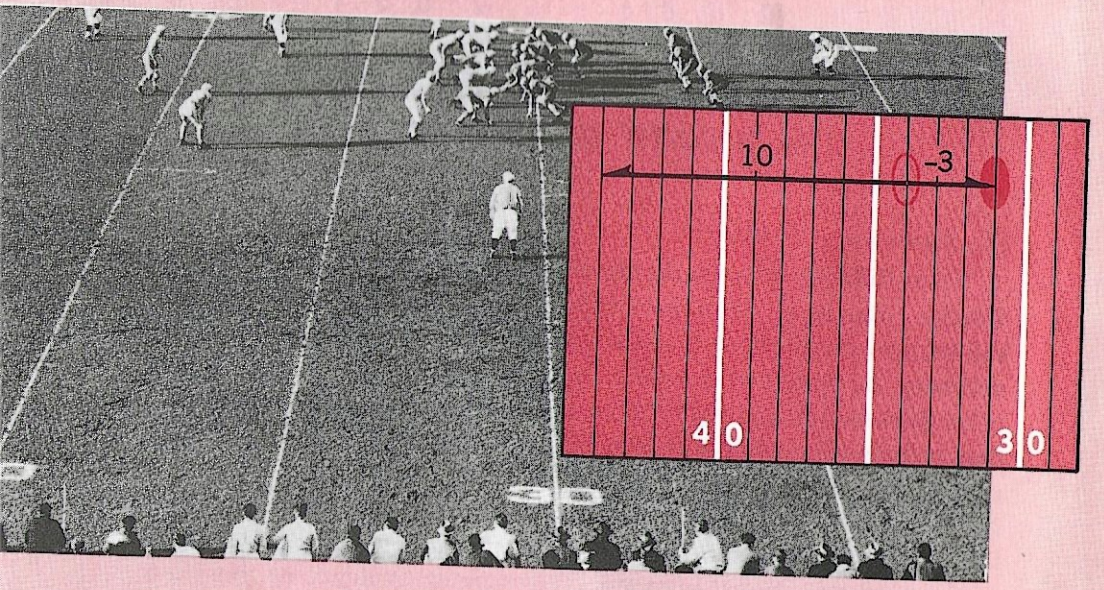
The Invention of Numbers

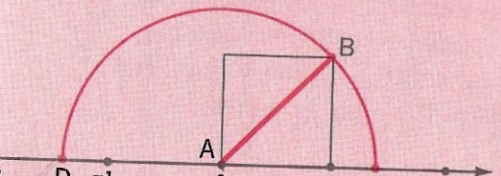
It is difficult to trace the development of mathematics before the Greeks. But as all primitive peoples today have some conception of the natural numbers, it is assumed that early man did also. Out of necessity, he would have wanted to keep a record of his animals and other possessions and used the idea of a one-to-one correspondence to match them with a handful of pebbles or a string of beads. Then the natural numbers would have logically developed. But as society changed from one of hunters to one of farmers and traders, numbers for parts of things were needed and this probably marked the invention of the positive rational numbers. No definite evidence of this exists, for the societies involved did not have writings. But since a number is an idea, not a symbol, numbers could be used before the development of numerals.

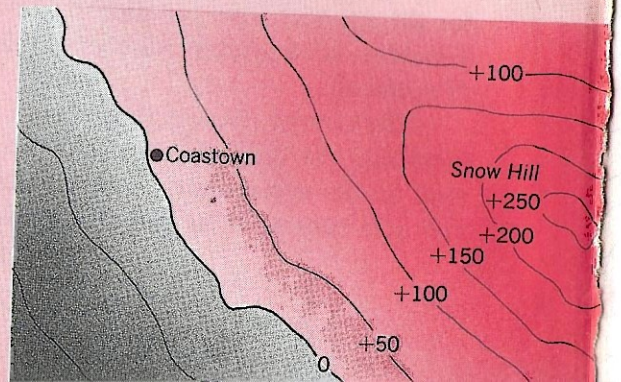
The Greeks discovered the irrational numbers in connection with measuring the diagonal of a square but refused to call them numbers. And in the seventeenth century, what are called imaginary numbers met with resistance, as had all extensions of the number system.











CHAPTER 2

THE SET OF REAL NUMBERS

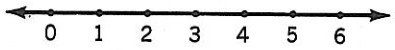
Not every problem can be solved with the numbers you have used so far in this book. You must learn how to operate with a more general system of numbers to progress very far in algebra. You must also learn other interpretations for the numbers that you already know. The following questions give some ideas about the real numbers.

- You have used 0 to mean "not any." However, does 0 on the thermometer mean that there is no temperature or no heat? On the thermometer, 0 matches a point that represents a temperature. With this reference point, you may measure temperature as degrees *above zero* or *below zero*. On the enlarged portion of the thermometer, which number matches A? B? What are the numbers called?
- Is the use of zero on the thermometer similar to the use of zero on a number line? To indicate direction from zero, you use the words *positive* and *negative*. Is *positive* up or down? right or left? Is *negative* up or down? right or left?
- On the number line in the lower left corner, $\overline{AC}$ and $\overline{AD}$ have the same length as $\overline{AB}$. What number matches point C? point D? What *kind* of number matches point C? point D?
- On the map, what do the plus and minus signs indicate?
- In football, there are gains of so many yards and losses of so many yards. How many yards must a team travel to have a first down? What is the reference point from which the yards are measured? On the football field shown, how many yards must the team travel to make a first down? What number must be added to $^{-}3$ so the sum is $^{+}10$? How can you use directed line segments to find the sum?
- How do you add, subtract, multiply, or divide any two positive real numbers? any two negative real numbers? a positive and a negative real number?

2–1 The Real Numbers

A set of points is often used as a geometric model for a number system. The diagrams below show certain sets of numbers matched with sets of points on the line. Answer the questions.

Whole Numbers

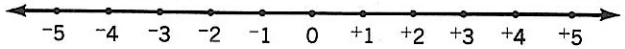


a. How do you locate the point for 10? for 99? for 100?

b. Are any points remaining on the line after each whole number is matched with a point?

Just as there are numbers "below zero" on a thermometer, there are also numbers matched with the points on a line to the left of 0.

Integers



c. How do you locate the point for $^+10$? for $^+25$? $^-10$? $^-25$?

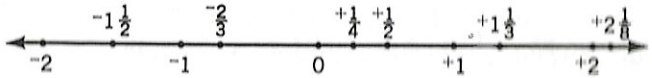
The raised signs are used to indicate whether the number is a positive integer or a negative integer. The number $^+10$ is named **positive** 10; the number $^-10$ is named **negative** 10. The set of **integers** includes the positive and the negative integers and 0. Often the raised sign $^+$ is omitted in designating a positive number. The symbol 25 also means *positive 25*.

$$^+25 = 25$$

When no sign is attached to a numeral, the number named by the numeral is positive or it is 0.

d. Are any points remaining on the line after each integer is matched with a point?

Rational Numbers



e. Where is the point for $^+\frac{1}{6}$ located? for $^-\frac{1}{6}$? $62\frac{1}{6}$? $^-62\frac{1}{6}$?

f. If a, b, and c are natural numbers and if $\frac{b}{c} < 1$, where do you locate the point for $\left(a + \frac{b}{c}\right)$? for $^-\left(a + \frac{b}{c}\right)$?

The raised signs indicate positive and negative rational numbers. Just as for the integers, if there is no sign, the number is positive. The set of **rational numbers** includes numbers of the form $\frac{a}{b}$, where a is any integer and b is any nonzero integer.

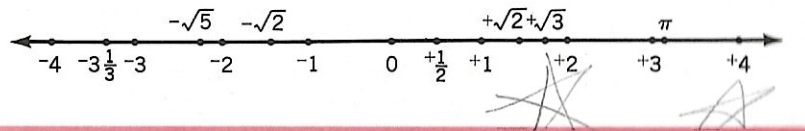
g. Are any points remaining on the line after each rational number is matched with a point?

The correct answer to **g** is that there are more points on the line than can be matched with the rational numbers. The members in the following set are not rational. They are *irrational numbers*.

$$\{\sqrt{2}, {}^-\sqrt{2}, \sqrt{3}, {}^-\sqrt{3}, \sqrt{5}, \pi, {}^-\pi, \sqrt[3]{72}\}$$

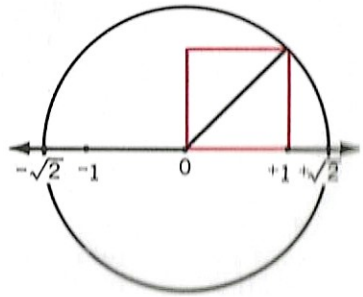
The union of the set of rational numbers and the set of irrational numbers is the set of **real numbers**. If you ask "Are there any points left on the line after matching each *real* number with a point?" the answer is "No." The irrational numbers are just those numbers that can be matched with the points on the line left over after the "rational points" are removed. Each point on the number line is matched with a number from the set of real numbers.

The relationship between real numbers and points on a line is an important assumption of algebra. It is Postulate 1 of Chapter 2.

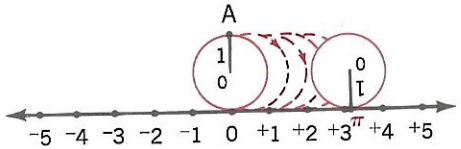


Postulate 2–1 There is a one-to-one correspondence between points on a line and the set of real numbers.

The point for $\sqrt{2}$ can be found by using the diagonal of the **unit square** (a square whose sides each measure one unit) as the radius of a circle. With 0 as the center, the points of intersection of the circle and the number line locate $^+\sqrt{2}$ and $^-\sqrt{2}$.

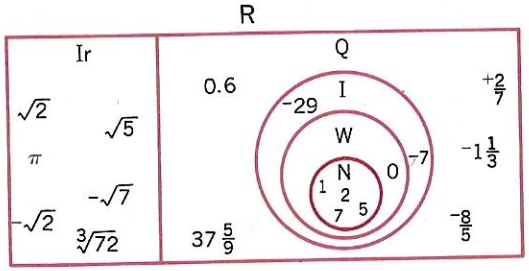


The point for π can be located by "rolling" a **unit circle** (a circle whose radius measures one unit) along the line, beginning at 0. After one half revolution to the right, as shown below, point A will match π. How could the point for $-\pi$ be located on the number line?



To summarize, the set of real numbers includes all the numbers you have studied so far. The diagram below shows some important relationships between subsets of real numbers.

R = The set of real numbers.
Ir = The set of irrational numbers.
Q = The set of rational numbers.
I = The set of integers.
W = The set of whole numbers.
N = The set of natural numbers.



Now try these

— Use the diagram that shows the subsets of the real numbers and tell whether each sentence is *True* or *False*.

1. $\sqrt{2} \in R$	**2.** $\sqrt{2} \in Q$	**3.** $\sqrt{2} \in I$	**4.** $\sqrt{2} \in W$
5. $-9 \in R$	**6.** $-9 \in Q$	**7.** $-9 \in I$	**8.** $-9 \in W$
9. $17 \in R$	**10.** $17 \in Q$	**11.** $17 \in Ir$	**12.** $17 \in N$
13. $Ir \subseteq R$	**14.** $R \subseteq Ir$	**15.** $Q \subseteq Ir$	**16.** $Q \subseteq R$
17. $I \subseteq Q$	**18.** $I \subseteq Ir$	**19.** $N \subseteq I$	**20.** $N \subseteq W$

Answers: **1.** True. **2.** False. **3.** False. **4.** False. **5.** True. **6.** True. **7.** True. **8.** False. **9.** True. **10.** True. **11.** False. **12.** True. **13.** True. **14.** False. **15.** False. **16.** True. **17.** True. **18.** False. **19.** True. **20.** True.

Exercises

A — Write *True* or *False* to describe each sentence.

1. $7 \in R$	**2.** $-6\frac{1}{2} \in Q$	**3.** $-7\frac{1}{2} \in Ir$	**4.** $8\frac{1}{2} \notin Q$
5. $-\sqrt{3} \notin I$	**6.** $8 \in N$	**7.** $-100 \in W$	**8.** $-100 \in I$
9. $-100 \in Ir$	**10.** $-100 \in R$	**11.** $\pi \in Ir$	**12.** $\pi \in Q$
13. $0 \in I$	**14.** $0 \in N$	**15.** $0 \in W$	**16.** $I \subseteq N$
17. $R \subseteq Q$	**18.** $Q \subseteq I$	**19.** $W \subseteq R$	**20.** $Ir \subseteq Q$
21. $W \subseteq Q$	**22.** $Q \subseteq W$	**23.** $R \subseteq I$	**24.** $N \subseteq Ir$

25. All rational numbers are real numbers.
26. All real numbers are rational numbers.
27. Some real numbers are rational numbers.
28. All integers are rational numbers.
29. No rational number is also an irrational number.
30. There exists a whole number that is not a natural number.

31. Rewrite the definition of $a > b$ on page 21 so that it is true for all real numbers.

— Using the letters N, W, I, Q, Ir, and R to stand for the sets of numbers, classify each of the following numbers. Some may belong to more than one set.

32. 28	**33.** 0
34. π	**35.** $0.\overline{15}$
36. 2.6738	**37.** -2
38. 25	**39.** $\frac{22}{7}$
40. $\frac{10}{5}$	**41.** $-3\frac{1}{3}$

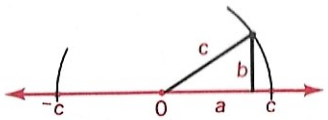
42. If an elevation on a government map is labeled -97 to indicate that a point is 97 feet below sea level, how would you describe a point that is labeled $+203$ on the same map?

43. Inspectors used to mark packages $+2$ to indicate that the package was two pounds over the amount stated on the bill. How do you think they marked a package that was a half pound underweight? (Some historians think that the marks used by such inspectors were the actual origin of the present plus and minus signs.)

44. In launching a satellite, the time 2 minutes, or 120 seconds, before liftoff is known as -120. What is the time $1\frac{1}{2}$ minutes after liftoff called?

B — Use the Pythagorean theorem to find the length of the hypotenuse, c, of a right triangle. Make a drawing of each triangle as shown. Use a compass to locate points on the line for c and ^{-}c.

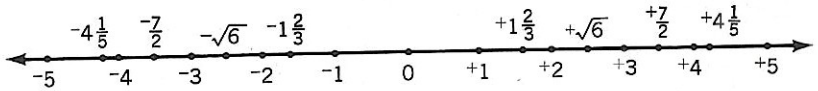
	a	b	c
45.	1	1	?
46.	3	4	?
47.	2	1	?
48.	7	8	?



C **49.** Which values for c in Exercises 45–48 are rational and which are irrational? Find other values for a and b that will give rational values for c.

2–2 Negatives and Absolute Value

On the number line below, the numbers $^{+}3$ and $^{-}3$ have something in common, as do $^{-}4\frac{1}{5}$ and $^{+}4\frac{1}{5}$. Can you identify the common characteristic of the numbers in such pairs?



The real numbers $^{+}3$ and $^{-}3$ are on opposite sides of 0, both the same distance from 0.

Definition Two distinct numbers that may be matched with points on the number line on opposite sides of 0, both the same distance from 0, are called opposites of each other. Another name for opposite of is negative of. The number 0 is its own opposite.

The dash – in front of a numeral or an expression means **the negative of.** For example, $-(^{+}3\frac{1}{4})$ means the negative of $^{+}3\frac{1}{4}$ and is read "the negative of positive three and one fourth." The dash means "negative of," but the *raised* dash means just "negative." Since $^{-}3\frac{1}{4}$ is the negative of $^{+}3\frac{1}{4}$, you can write the following.

$$-(^{+}3\tfrac{1}{4}) = {}^{-}3\tfrac{1}{4}$$

Read the following sentences and tell why each is true.

a. $-(^{+}1\frac{3}{8}) = {}^{-}1\frac{3}{8}$ **b.** $-(^{-}4) = {}^{+}4$ **c.** $^{-}2 = -(2)$

d. $-(\frac{7}{9}) = {}^{-}\frac{7}{9}$ **e.** $-(^{+}\sqrt{6}) = {}^{-}\sqrt{6}$ **f.** $-0 = 0$

If x represents a positive real number, does $-x$ represent a positive or a negative number? If x represents a negative number, is $-x$ positive or negative?

The numbers $^{-}5$ and $^{+}5$ are negatives of each other. Each is 5 units from zero. The number of units a number is from zero, disregarding direction, is called the **absolute value** of the number. The absolute value of $^{-}5$ is 5; also, the absolute value of $^{+}5$ is 5. Two vertical marks, | |, indicate absolute value. You write $|\,{}^{-}5\,|$ to mean the absolute value of $^{-}5$. Thus, the sentence $|\,{}^{-}5\,| = 5$ is true. Read the following sentences and tell why each is true.

g. $|\,{}^{-}6\frac{1}{2}\,| = {}^{+}6\frac{1}{2}$, or $6\frac{1}{2}$ **h.** $|\,{}^{+}4\frac{1}{4}\,| = {}^{+}4\frac{1}{4}$, or $4\frac{1}{4}$

i. $|\,100\,| = {}^{+}100$, or 100 **j.** $|\,0\,| = 0$

The definition for $|\,x\,|$ must be stated so that $|\,x\,|$ names a positive number when $x < 0$ or when $x > 0$.

Definition The absolute value of a real number x, symbolized by $|\,x\,|$, is the greater of the two numbers x and $-x$, $x \neq 0$. If x is 0, then $|\,x\,|$ is also 0.

How does the definition assure you that $|\,x\,|$ is positive when $x < 0$? when $x > 0$? Use the definition to explain **g–j** above.

Checkpoint

1. What is the negative of a given real number?
2. What symbol denotes *is the negative of*?
3. What is meant by *the absolute value of a real number*? What symbol denotes absolute value?

Exercises

A — What is the negative of each real number?

1. $^{-}3$	**2.** $^{+}\frac{1}{2}$	**3.** $^{+}0.321$	**4.** $^{-}\frac{0}{2}$
5. $\sqrt{7}$	**6.** 0	**7.** $^{-}\pi$	**8.** $^{+}12{,}597$

What is the absolute value of each real number?

9. $^+6$	**10.** $^-6$	**11.** $^-3\frac{1}{2}$	**12.** $^-\pi$
13. $^-19.5$	**14.** 15	**15.** $^-25$	**16.** 0
17. $^-16$	**18.** $^-100$	**19.** 695	**20.** $^-462$
21. $^-72.5$	**22.** 22,000	**23.** $^+37.75$	**24.** $^-7.25$
25. $^-\sqrt{2}$	**26.** $\sqrt{5}$	**27.** 86.75	**28.** $^+\sqrt{7}$

29. What are the two numbers that have an absolute value of 10? Is there a number that has an absolute value of $^-10$? Explain.

Write *True* or *False* to describe each of the following sentences.

30. $\lvert {}^+8 \rvert = 8$	**31.** $\lvert {}^-231 \rvert = 231$	**32.** $\lvert {}^-663 \rvert = \lvert 663 \rvert$
33. $\lvert {}^+9 \rvert > 9$	**34.** $\lvert {}^-235 \rvert > \lvert {}^+235 \rvert$	**35.** $\lvert {}^-35 \rvert > {}^-35$
36. $\lvert {}^-27 \rvert < 17$	**37.** $\lvert 17.9 \rvert < \lvert {}^-17.9 \rvert$	**38.** $\lvert {}^-0.01 \rvert < \lvert {}^-0.001 \rvert$
39. $\lvert {}^-\frac{1}{3} \rvert < \frac{1}{4}$	**40.** $\frac{2}{3} < \lvert {}^-\frac{3}{4} \rvert$	**41.** $\lvert \frac{1}{8} \rvert < {}^-\frac{1}{7}$

Rename each number without using the signs for "negative of" or "absolute value of."

42. $-({}^+5)$	**43.** $-({}^-9)$	**44.** -3.14	**45.** $-({}^-3.14)$
46. $-\lvert {}^+5 \rvert$	**47.** $-\lvert {}^-9 \rvert$	**48.** $-\lvert 3.14 \rvert$	**49.** $-\lvert -3.14 \rvert$

Replace each ● with $<$, $>$, or $=$ to make true sentences.

50. $\lvert {}^+0.23 \rvert$ ● $\lvert {}^-3.7 \rvert$	**51.** $\lvert {}^-6 \rvert$ ● $\lvert {}^+6 \rvert$	**52.** $\lvert {}^-\frac{1}{2} \rvert$ ● $\lvert {}^+\frac{1}{3} \rvert$
53. $\lvert {}^+0.01 \rvert$ ● $\lvert {}^-0.001 \rvert$	**54.** $\lvert 0 \rvert$ ● $\lvert {}^-1 \rvert$	**55.** $\lvert {}^-\frac{1}{6} \rvert$ ● $\lvert {}^+\frac{1}{7} \rvert$
56. $\lvert {}^-1 \rvert$ ● $\lvert {}^+1 \rvert$	**57.** $\lvert 0 \rvert$ ● $-\lvert 0 \rvert$	**58.** $\lvert {}^-3 \rvert$ ● $\lvert {}^-\frac{1}{4} \rvert$
59. $\lvert {}^-3 \rvert$ ● $\lvert {}^+4 \rvert$	**60.** $\lvert -2 \rvert$ ● $\lvert -2 \rvert$	**61.** $\lvert -5 \rvert$ ● $\lvert {}^-5 \rvert$

Find the real number that makes each sentence true.

62. $\lvert {}^-2 \rvert = x$	**63.** $\lvert {}^-7 \rvert = a$	**64.** $\lvert {}^+4.6 \rvert = m$
65. $\lvert {}^-2\frac{2}{3} \rvert = y$	**66.** $-\lvert 9 \rvert = b$	**67.** $a = \lvert {}^-96 \rvert$
68. $-({}^-3) = x$	**69.** $-({}^-9) = y$	**70.** $-({}^+3) = m$
71. $a = -({}^-9\frac{1}{2})$	**72.** $q = -(0)$	**73.** $-a = {}^+6$
74. $-m = {}^-6$	**75.** $-a = 8$	**76.** $-m = {}^-14\frac{1}{5}$

B Is each sentence in Exercises 77–91 *True* or *False*?

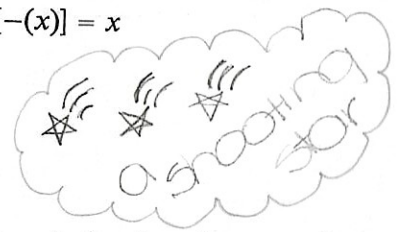
R = The set of real numbers.
P = The set of positive real numbers.
I = The set of integers.
Ne = The set of negative real numbers.

77. $I \subseteq R$	**78.** $R \nsubseteq I$	**79.** $P \nsubseteq R$
80. $Ne \subseteq R$	**81.** $^+4\frac{1}{2} \in P$	**82.** $^-9\frac{1}{2} \in Ne$

83. $6\frac{2}{3} \notin R$	**84.** $0 \in P$	**85.** $0 \notin Ne$
86. $0 \in R$	**87.** $-({}^-6) \in P$	**88.** $-({}^+6\frac{1}{2}) \in P$
89. $-({}^-96) \notin Ne$	**90.** $-({}^-6\frac{1}{4}) \in I$	**91.** $Ne \subseteq I$

C **92.** Is the following sentence true? Why or why not?

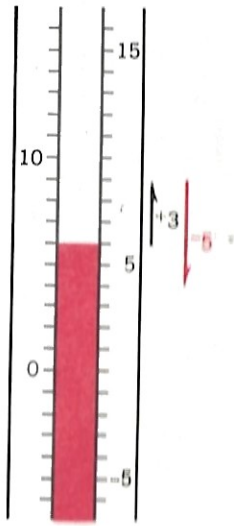
$\forall x \in R \quad -[-(x)] = x$



2–3 Vectors

You have been using real numbers to locate points on a line with reference to a point labeled 0. Numbers such as $^+3$ and $^-3$ can describe temperatures above or below zero.

There is another useful model for the real numbers. It uses the notion of "change." The number $^+3$ denotes a *rise* of 3° in temperature or a change of $^+3$. The real number $^-5$ denotes a drop of 5° or a change of $^-5$. The real numbers $^+3$ and $^-5$ denote both the amount of change and the direction of the change. Thus, if the temperature is 6°, as in the diagram, and there are changes of $^+3$ and $^-5$, what is the final temperature?



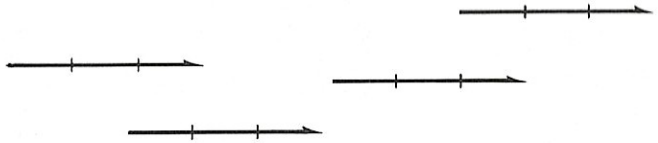
The geometric models for changes of $^+3$ and $^-5$ shown in the diagram are **vectors.** In the figure the black line segment marked with a half arrowhead is the vector $^+3$. The vector for $^-5$ is shown in red.

Every line segment has two endpoints. For a vector, one endpoint is called the **initial point** and the other is called the **terminal point.** A vector has direction, whereas a line segment has no direction. The half arrowhead shows the terminal point of the vector, indicating its direction. To identify a vector m, a half arrowhead is placed over the variable: $\vec{m}$.

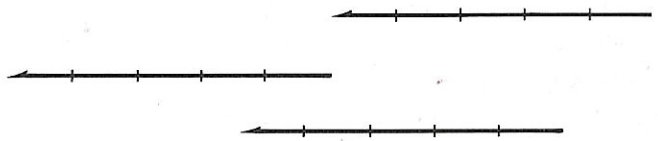
Definition A vector is a directed line segment.

Where does the vector that shows $^+3$ begin? Where does it end? Is the direction up or down? Where does the vector that shows $^-5$ begin? Where does it end? What is its direction?

All the vectors below show a change of $+3$, a change of 3 to the right.

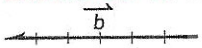


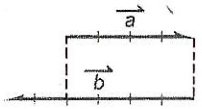
The vectors below show a change of -5, a change of 5 to the left.



These drawings show that the change can be from any beginning position.

To find the sum of two vectors, think of combining the two changes into one change.

EXAMPLE 1. Draw a vector that is the sum of and  .



① Begin with $\vec{a}$. It shows a change of 4 units to the right.

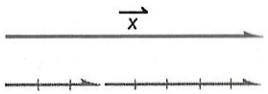
② Think of $\vec{b}$ as starting at the end of $\vec{a}$. It shows a change of 6 units to the left.

③ The sum is the vector that shows the total change. The total change is 2 units to the left.

Do you get the same result if you start with $\vec{b}$?

EXAMPLE 2. Draw a vector that is the sum of the two vectors shown.

Place the initial point of the second at the terminal point of the first. The sum is the vector shown in black.



The first vector shows a change of 3 units to the right; the second shows a change of 5 more units to the right. Thus, the sum, marked $\vec{x}$ to show it is a vector, shows a total change of 8 units to the right.

Checkpoint

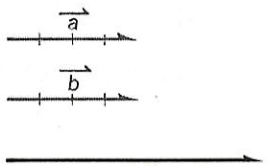
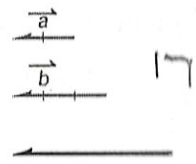
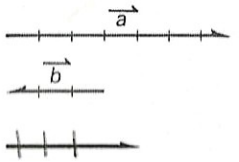
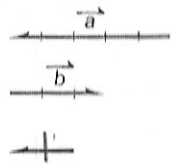
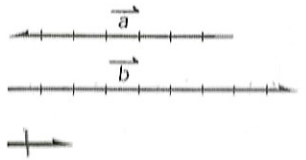
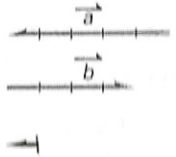
1. What is the difference between a line segment and a vector?
2. How do you draw a vector that is a model for $+7$? for -10? for a if $a > 0$? for $-a$ if $a > 0$?

Exercises

A

1. If $+3$ represents a gain of \$3, ? represents a loss of \$5.
2. If $+\frac{3}{4}$ on a chart indicates that a stock on the stock market increased in value by \$.75, what does $-\frac{1}{2}$ mean on the same chart?
3. Could the area of a rectangular region be represented by a negative number? Could the change in area of a rectangular region when one dimension is changed be represented by a negative number?
4. Would you use a positive number or a negative number to record the altitude of a landmark that is two hundred feet below sea level? How would you record the change in altitude when traveling from that landmark to a lake that is one hundred feet below sea level?
5. Positive and negative numbers are sometimes called *directed numbers*. Can you explain why this name might be used?

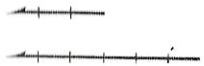
The vectors $\vec{a}$ and $\vec{b}$ in the exercises below represent two vectors to be added. The vector shown in black is the sum of $\vec{a}$ and $\vec{b}$. What is the length and direction of each vector?

6. 
7. 
8. 
9. 
10. 
11. 

— Draw a vector that is the sum of the two shown. Give the real number that is associated with each of the three vectors.

12.

13.

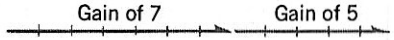
14. 

15.

16.

17.

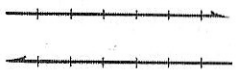
18. The drawing shows the results of two football plays.
One play: Gain of 7 yards.
Next play: Gain of 5 yards.



What is the result of the two plays?

What do you need to know to be sure the team made a first down?

19. The two vectors show a gain of 7 and a loss of 7. What is the sum of the two vectors?



— Draw vectors to find the result of the following.

20. Gain of 7; then a loss of 4.
21. Loss of $6\frac{1}{2}$; then a loss of $7\frac{1}{2}$.
22. Gain of $8\frac{3}{4}$; then a gain of $7\frac{1}{4}$.
23. Loss of 5; then a gain of 9.
24. Increase of 8; then a decrease of 17.
25. Decrease of 6; then a decrease of $3\frac{1}{2}$.

— What real number is associated with each vector described below?

26. Begins at 0 and ends at $+3$.
27. Begins at 0 and ends at -3.
28. Begins at 0 and ends at $-2\frac{1}{2}$.
29. Begins at 0 and ends at $-\sqrt{2}$.
30. Begins at 0 and ends at $+100$.
31. Begins at 0 and ends at -100.
32. Begins at 0 and ends at -17.
33. Begins at 0 and ends at $-16\frac{5}{9}$.
34. Repeat Exercises 26–33, but start each vector at -2.

B — If vector $\vec{a}$ is 9 units long and vector $\vec{b}$ is 7 units long, what is the sum of the two vectors if the following is true?

35. Both $\vec{a}$ and $\vec{b}$ are *to the left.*
36. Both $\vec{a}$ and $\vec{b}$ are *to the right.*
37. $\vec{a}$ is *to the left* and $\vec{b}$ is *to the right.*
38. $\vec{b}$ is *to the left* and $\vec{a}$ is *to the right.*

C — The length of vector a is represented as $|\vec{a}|$. For example, the length of the vector -8 is $|-8|$, or 8. What is the sum of $\vec{a}$ and $\vec{b}$ when the following conditions hold?

39. Both are *to the left.*
40. Both are *to the right.*
41. $\vec{a}$ is *to the left,* $\vec{b}$ is *to the right,* and $|\vec{a}| > |\vec{b}|$.
42. $\vec{a}$ is *to the left,* $\vec{b}$ is *to the right,* and $|\vec{a}| < |\vec{b}|$.
43. $\vec{a}$ is *to the right,* $\vec{b}$ is *to the left,* and $|\vec{a}| > |\vec{b}|$.
44. $\vec{a}$ is *to the right,* $\vec{b}$ is *to the left,* and $|\vec{a}| < |\vec{b}|$.

2–4 Postulates for Addition of Real Numbers

The set of real numbers is a mathematical model of the set of vectors. You know how to add vectors. If the real numbers are to be an accurate model, the **sum of two real numbers** should be defined so that it corresponds to the sum of the two matching vectors. Experience with vectors should convince you that when addition is defined in this way, the real numbers have the properties of the fractional numbers, which were listed on page 31. You assume that this is true. Four postulates for addition of real numbers are summarized below.

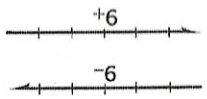
	Postulates for Addition of Real Numbers	Name
2–2	$\forall x \in R \quad \forall y \in R \quad x + y$ **is one and only one real number.**	Closure postulate.
2–3	$\forall x \in R \quad \forall y \in R \quad x + y = y + x$	Commutative postulate.
2–4	$\forall x \in R \quad \forall y \in R \quad \forall z \in R$ $(x + y) + z = x + (y + z)$	Associative postulate.
2–5	$\forall x \in R \quad x + 0 = 0 + x = x$, **and 0 is the only real number with this property.**	Identity postulate.

Besides the given four postulates for addition of real numbers, one more can be stated. The example below illustrates Postulate 2–6.

EXAMPLE. What number added to $^+6$ gives 0?

Think of $^+6$ as a "gain of 6." Then a "loss of 6" puts you back at 0. Thus, adding $^-6$ to $^+6$ gives a sum of 0. Are $^+6$ and $^-6$ opposites of each other?

$$\begin{array}{r} {}^+6 \\ +\ {}^-6 \\ \hline 0 \end{array}$$



Could you answer the question in the example if the number had been $^+9$? if it had been $^-3$? if it had been x? For each real number x, there is another real number $-x$, the negative of x, such that their sum is 0. The sentence $x + (-x) = 0$ is always true. This is Postulate 2–6.

Postulates for Addition of Real Numbers	Name
2–6 $\forall x \in R$ **there exists** $-x \in R$ **such that** $x + (-x) = 0$ **is true. For each** x **there is one and only one** $-x$.	Additive inverse postulate.

If two numbers such as $\frac{3}{2}$ and $\frac{2}{3}$ are *multiplicative inverses*, then their product is 1: $\frac{3}{2} \times \frac{2}{3} = 1$. If two numbers are *additive inverses*, what is their sum?

You can simplify the way you write the numerals for the real numbers by writing 6 for $^+6$ in most cases. Similarly, since the negative of 6 is shown by the symbol -6, you may write -6 instead of $^-6$. Tell why these sentences are true.

a. $-7\frac{1}{2} = {}^-7\frac{1}{2}$ **b.** $-\sqrt{2} = {}^-\sqrt{2}$ **c.** $-100 = {}^-100$

In general, if a is a positive number, $-a$ is a negative number. In fact, the *negative of a* when a is positive is equal to the number *negative a*. Hereafter, the sign for "negative of" will be used to indicate negative numbers. You may read -6 usually as "negative 6," although in some circumstances it may be more appropriate to say "the negative of 6." Similarly, $+6$ will sometimes be used to mean $^+6$.

Checkpoint

1. Is the negative of a real number always a negative number? Explain.

2. Explain why any negative number may be written as the negative of a positive number.

3. Explain why the set of real numbers is closed under addition.

Exercises

A — If the sentence is true, give the name of the postulate that it illustrates. If the sentence is false, write a true sentence by changing one of the real numbers, and then name the postulate. The sign for addition is in color to remind you that it is not the sign for a positive number.

1. $6 + (-2) = (-2) + 6$
2. $3 + (-3) = 0$
3. $(3 + 4) + (-5) = 3 + [4 + (-5)]$
4. $-11 + 0 = -11$
5. $-9 + (-11) = 11 + (-9)$
6. $-27 + 27 = 0$
7. $(5 + 9) + (-8) = (-8) + (5 + 9)$
8. $0 + (-5) = -5$
9. $-\pi + \pi = 0$
10. $-\sqrt{2} + \sqrt{2} = 0$

— Find a real number that will replace x so that the sentence is true.

11. $-4 + 3 = x + (-4)$
12. $5 + x = 0$
13. $(7 + x) + (-9) = 7 + [(-2) + (-9)]$
14. $x + (-3) = 0$
15. $9 + (-9) = x$
16. $-11 + x = -11$
17. $x + 0 = -29$
18. $-44 + x = 0$
19. $x + \frac{3}{4} = 0$
20. $-0.2 + x = 0$

B — Which postulate for addition of real numbers is a mathematical model of the given situation in Exercises 21–23? Write a mathematical sentence to illustrate each.

21. Two hours later than 11 A.M. is the same time as eleven hours later than 2 A.M.

22. A party of mountain climbers traveled the same distance down the mountain as they did in climbing to its peak, and they arrived where they had started. The mountain peak was 5247 feet above their camp.

23. The price of stock in the Acme Apex Corporation dropped an eighth of a dollar on Monday and it stayed at that level on Tuesday. The net change over the two days was $-\frac{1}{8}$.

24. Make up your own examples of physical situations for which Postulates 2–3, 2–4, and 2–5 for addition might serve as the mathematical models.

2–5 Addition of Real Numbers

Negative numbers often occur in applications. For example, if a plane is traveling 400 miles an hour into a head wind of 80 miles an hour, the ground speed of the plane is $400 + (-80)$. To solve such a problem and find the answer as 320, you need to know how to add positive and negative numbers.

The examples below show how the postulates of addition can be used to find sums. The sign for addition, +, is written in color for a while to stress that it is the sign of the operation.

Sum of Two Negative Numbers

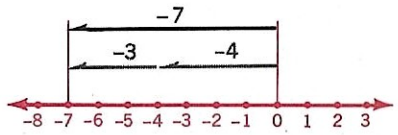
EXAMPLE 1. Find the sum of -4 and -3.

Think of -4 as a "loss of 4" and of -3 as a "loss of 3," as shown in the vector diagram.

Loss of $(4 + 3)$

$\downarrow$

$-(4 + 3)$, or -7



The sum of -4 and -3 is -7.

Draw vector diagrams for each true sentence below. Can you discover a pattern?

a. $-5 + (-7) = -12$ **b.** $-6\frac{1}{2} + (-8\frac{1}{2}) = -15$

c. $-11 + (-8) = -19$ **d.** $-5.2 + (-8.1) = -13.3$

e. $-15 + (-20) = -35$ **f.** $-4 + (-99) = -103$

Now try to state a rule for adding two negative numbers.

You can prove the rule for adding any two negative numbers by using the postulates for addition, as outlined in Exercise 77 on pages 70 and 71. The proof is left as an exercise. A statement proved from postulates and definitions is a **theorem.**

Theorem 2–1 The sum of two negative numbers is a negative number. The sum is the negative of the sum of their absolute values.

If $-a < 0$ and $-b < 0$, then $(-a) + (-b) = -(|a| + |b|)$.

Use the theorem to check sentences **a–f** above. Then see if you can find the numbers to replace x and make sentences **g–j** true.

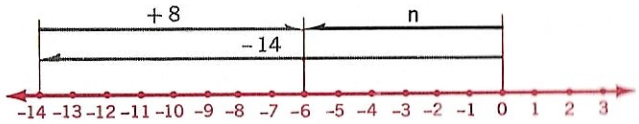
g. $(-12) + (-23) = x$ **h.** $(-6.1) + (-6) = x$

i. $(-8\frac{1}{2}) + (-8\frac{1}{2}) = x$ **j.** $(-325) + (-163) = x$

Sum of a Positive Number and a Negative Number

EXAMPLE 2. Find n if $-14 + 8 = n$ is true.

One way is to use the vector diagram below. A vector for -14 begins at 0 and extends 14 units to the left. Then a vector for $+8$ is drawn, beginning at the terminal point of the vector for -14.



The sum is n, the total change. So n is -6.

Here is another way to find the sum. Begin by thinking of -14 as $-6 + (-8)$ so that you can use the additive inverse postulate.

	Statements	*Reasons*
1.	$-14 + 8 = [-6 + (-8)] + 8$	1. Theorem 2–1.
2.	$= -6 + (-8 + 8)$	2. Associative postulate.
3.	$= -6 + 0$	3. Additive inverse postulate.
4.	$= -6$	4. Identity postulate.

The sum, n, is -6.

The outline above shows how you can think of the 8 matching with part of the -14, leaving -6 in the part not matched.

Verify each of the following, using either method in Example 2.

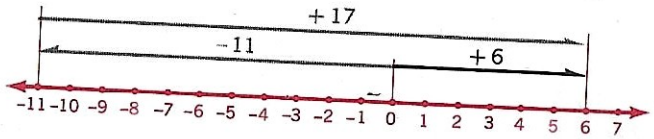
k. $-16 + 5 = -11$ **l.** $-12 + 9 = -3$ **m.** $-50 + 8 = -42$

n. $-7 + 4\frac{1}{2} = -3\frac{1}{2}$ **o.** $16 + (-17) = -1$ **p.** $20.4 + (-31.4) = -11$

If the absolute value of the negative number is greater than the absolute value of the positive number, is the sum a positive or a negative number?

EXAMPLE 3. Find n if $-11 + 17 = n$ is true.

The diagram shows one way. You can think of a "loss of 11" and a "gain of 17." The result is a "gain of 6."



Here is another way.

	Statements	*Reasons*
1.	$-11 + 17 = -11 + (11 + 6)$	1. Renaming 17 as $11 + 6$.
2.	$= (-11 + 11) + 6$	2. Associative postulate.
3.	$= 0 + 6$	3. Additive inverse postulate.
4.	$= 6$	4. Identity postulate.

Verify each of the following, using either method in Example 3.

q. $12 + (-7) = 5$ **r.** $45 + (-35) = 10$ **s.** $65 + (-52) = 13$

t. $-2\frac{1}{2} + 4\frac{1}{2} = 2$ **u.** $-1.2 + 7.2 = 6$ **v.** $-4.2 + 15 = 10.8$

If the absolute value of the positive number is greater than the absolute value of the negative number, is the sum a positive or a negative number?

Now try to make a rule for adding any positive and any negative number.

Here is the way that one algebra student made a generalization. To add a positive and a negative number,

① find the absolute value of each number and subtract the smaller absolute value from the greater;

② if the absolute value of the positive number is greater, the sum is the result in step 1;

③ if the absolute value of the negative number is greater, the sum is the negative of the result in step 1.

This generalization is Theorem 2–2. In algebraic symbols, it is stated as follows.

Theorem 2–2 If $x < 0$ and $y > 0$, then $x + y = -(|x| - y)$ when $|x| > y$, and $x + y = y - |x|$ when $|x| < y$.

Now try these

Find the sums.

1. $-3 + (-4)$ **2.** $-9 + (-5)$ **3.** $-3 + 4$

4. $3 + (-4)$ **5.** $-9 + 5$ **6.** $9 + (-5)$

7. $3 + 4$ **8.** $-12 + 18$ **9.** $5 + (-5)$

Answers: **1.** -7 **2.** -14 **3.** 1 **4.** -1 **5.** -4 **6.** 4 **7.** 7 **8.** 6 **9.** 0

Exercises

A Name each sum with a single numeral.

1. $-3 + (6)$ **2.** $3 + 6$ **3.** $(-3) + (-6)$

4. $3 + (-6)$ **5.** $9 + (-6)$ **6.** $(-9) + 6$

7. $2 + (-5)$ **8.** $(-7) + (-2)$ **9.** $(-4) + (-9)$

10. $7 + 5$ **11.** $(-3) + 7$ **12.** $(-10) + (-3)$

13. $6 + (-2)$ **14.** $(-9) + (-6)$ **15.** $8 + (-3)$

16. $6 + 2$ **17.** $(-10) + (-4)$ **18.** $-8 + 3$

19. $9 + 0$ **20.** $0 + 9$ **21.** $(-9) + 8$

22. $7 + (-3)$ **23.** $8 + (-7)$ **24.** $(-3) + (-7)$

25. $9 + 6$ **26.** $(-6) + 3$ **27.** $5 + (-6)$

28. $(-3) + 7$ **29.** $(-9) + 0$ **30.** $0 + (-9)$

31. $2\frac{3}{4} + (-1\frac{1}{2})$ **32.** $(-\frac{3}{5}) + (-\frac{2}{3})$ **33.** $7 + 4\frac{1}{2}$

34. $-9\frac{1}{3} + 6\frac{3}{4}$ **35.** $(-6\frac{5}{12}) + 8\frac{1}{6}$ **36.** $8.6 + (-3.7)$

37. $(-4.5) + 2.4$ **38.** $8.0 + (-3.2)$ **39.** $(-6.0) + (-7.2)$

40. $8.2 + 6.4$ **41.** $1.3 + (-0.7)$ **42.** $2.4 + (-4.5)$

43. $(-3.7) + 8.6$ **44.** $(-4.0) + 0.7$ **45.** $(-1.6) + (-6.0)$

46. $[5 + (-8)] + [9 + (-2)]$ **47.** $[5 + (-7)] + [(-8) + (-3)]$

48. $[-8 + 9] + [7 + (-5)]$ **49.** $[7 + 4] + [(-9) + 8]$

50. $[-12 + (29)] + [37 + (-18)]$ **51.** $[(-15) + 2] + [37 + (-22)]$

52. Find the sum of the numbers named in each row and in each column.

							Row sums
	−3	7	−2	1	0	4	?
	−2	5	8	0	−3	7	?
	−9	3	5	6	−4	−10	?
	−6	−8	−6	−5	−5	9	?
	10	−7	6	7	4	−7	?
Column sums	?	?	?	?	?	?	?

Add the row sums and the column sums. The two results should be the same.

Evaluate each of the following expressions.

53. $|3| + |15|$ **54.** $|-3| + |-9|$ **55.** $|23| + |-18|$
56. $|-15| + |14|$ **57.** $|6| + |-8|$ **58.** $|-4| + |8|$
59. $|-9| + |9|$ **60.** $|9| + |-12|$ **61.** $|0| + |0|$
62. $|\frac{1}{2}| + |\frac{2}{3}|$ **63.** $|0.2| + |-0.37|$ **64.** $|-235| + |16|$

B Find the real number that can replace the variable to make a true sentence.

65. $x + 5 = 12$ **66.** $6 + y = -2$ **67.** $-8 + 22 = n$
68. $10 + t = 1$ **69.** $a + (-7) = -17$ **70.** $-14 + u = 42$
71. $7 + c = 0$ **72.** $m + 10 = 20$ **73.** $13 + z = 9$
74. $v + 14 = 2$ **75.** $16 + 18 = r$ **76.** $g + 7 = 3$

C **77.** Some of you may like to see how to prove that $-4 + (-3) = -(4 + 3)$ is true, using the postulates for addition. To prove it, you must show that $-4 + (-3)$ is the negative of $(4 + 3)$, or that the sum of $-4 + (-3)$ and $(4 + 3)$ is 0.

Complete the following by giving the reason why each statement is true.

	Statements	*Reasons*
1.	$[-4 + (-3)] + (4 + 3) = -4 + [(-3) + 4] + 3$	1. ?
2.	$= -4 + [4 + (-3)] + 3$	2. ?
3.	$= (-4 + 4) + [(-3) + 3]$	3. ?
4.	$= 0 + 0$	4. ?
5.	$= 0$	5. ?

Once you know that the sum is 0, you know that $-4 + (-3)$ and $(4 + 3)$ are inverses. What part of the additive inverse postulate tells you that if $-4 + (-3)$ is the negative of $(4 + 3)$, then $-4 + (-3)$ must be the same number as $-(4 + 3)$? Hence, $-4 + (-3) = -(4 + 3)$ is true.

Prove that each of the following sentences is true.

78. $-9 + (-15) = -24$ **79.** $-9 + 7 = -2$ **80.** $-9 + 17 = 8$

81. Prove that $-a + (-b) = -(a + b)$ is true $a \in R$, $b \in R$.

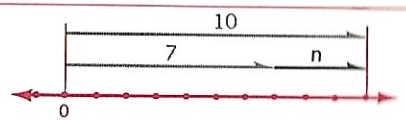
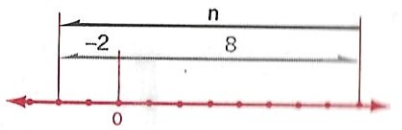
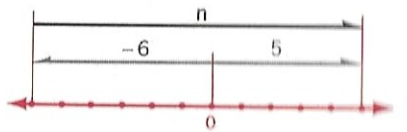
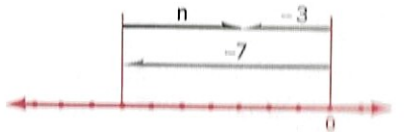
82. Prove that $-(-a) = a$ by using the additive inverse postulate.

2–6 Subtraction of Real Numbers

If you bought a shirt for \$2.49 and gave the salesman a five-dollar bill, he would give you your change (the difference between \$5.00 and \$2.49) probably by counting it out, saying, "\$2.49, \$2.50, \$2.75, \$3.00, \$4.00, \$5.00." What he has really done is added as follows.

$$2.49 + 0.01 + 0.25 + 0.25 + 1.00 + 1.00 = 5.00$$

In other words, to find \$5.00 − \$2.49, the salesman thought "\$2.49 *plus* some amount equals \$5.00."

Example	Vector diagram	Solution set
a. $10 - 7 = n$ Think: $7 + n = 10$		{3}
b. $(-2) - 8 = n$ Think: $8 + n = -2$		{−10}
c. $5 - (-6) = n$ Think: $-6 + n = 5$		{11}
d. $(-3) - (-7) = n$ Think: $-7 + n = -3$		{4}

These examples suggest the following definition of subtraction for real numbers a and b.

Definition To subtract b from a, written $a - b$, means to find the number c such that $b + c = a$.

$$\forall a \in R \quad \forall b \in R \quad a - b = c \text{ if and only if } b + c = a$$

Now try these

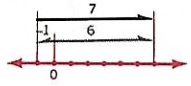
— Use the definition of subtraction to find the solution set for each sentence. The replacement set is the set of real numbers. Draw a vector diagram for each.

1. $9 - 3 = n$ **2.** $6 - (-1) = x$ **3.** $(-8) - 3 = n$

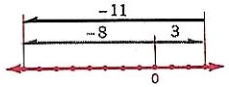
4. $(-9) - (-5) = m$ **5.** $(-18) - 13 = n$ **6.** $(-17) - 11 = y$

Answers: **1.** {6}

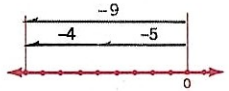
2. {7}



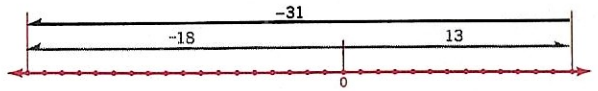
3. {−11}



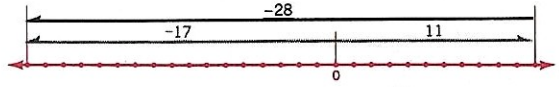
4. {−4}



5. {−31}



6. {−28}



Exercises

A — Use the definition of subtraction to find the solution set for each sentence. Draw vector diagrams for Exercises 1–5. The replacement set for each variable is the set of real numbers.

1. $-19 - (-10) = n$ **2.** $-17 - 8 = b$ **3.** $-15 - (-11) = a$

4. $13 - 6 = x$ **5.** $15 - (-9) = c$ **6.** $-18 - (-13) = t$

7. $7.5 - (-4.5) = u$ **8.** $-2\frac{1}{2} - 3\frac{1}{2} = y$ **9.** $-4.4 - (-3.6) = v$

10. $4.01 - 6.36 = f$ **11.** $-45 - 81 = r$ **12.** $-8\frac{1}{4} - (-12\frac{3}{4}) = g$

13. A football team gained 3 yards on one play and lost 5 yards on the next. What was the total change in position?

14. A stock lost $1\frac{1}{2}$ dollars on Wednesday and another $\frac{3}{4}$ dollar on Thursday. What was the total change for the two days?

15. A stock lost $1\frac{1}{4}$ dollars on Tuesday, and by the closing of the exchange on Wednesday, it was down $2\frac{3}{4}$ dollars from its opening on Tuesday morning. What happened on Wednesday?

16. An airplane traveling 385 miles per hour ran into a head wind of 72 mph. What was its speed with respect to the ground?

17. An airplane traveling 410 miles per hour found it had a ground speed of only 385 miles per hour. If it was traveling directly into the wind, how fast was the wind?

B **18.** A football team is on its own 12-yard line. On the first play, it gains 7 yards. After the second play, it is on its own 9-yard line. What happened during the second play?

19. The Texas Rio Molasses Corporation stock stood at $91\frac{1}{8}$ at the opening of trading on Monday. It lost $3\frac{1}{2}$ dollars on Monday, and by the closing Tuesday, it was at $90\frac{1}{4}$. What happened Tuesday?

2–7 Using the Postulates in Subtraction

The definition of subtraction may be used directly to solve simple problems, but for more complicated situations, you need a quicker way to deal with subtraction of real numbers. Have you noticed a way of simplifying subtraction by rewriting it as addition? You might be able to guess that the subtraction example $4 - (-7)$ has the same answer as the addition example $4 + 7$. Here is a way to show this is the case.

EXAMPLE. Show that $4 - (-7)$ is the same as $4 + 7$.

You can set your work up as a formal chain of statements.

	Statements	*Reasons*
1.	$4 - (-7) = n$	1. Given.
2.	$(-7) + n = 4$	2. Definition of subtraction.
3.	$7 + [(-7) + n] = 7 + 4$	3. Closure postulate.
4.	$[7 + (-7)] + n = 7 + 4$	4. Associative postulate.
5.	$0 + n = 7 + 4$	5. Additive inverse postulate.
6.	$n = 7 + 4$	6. Identity postulate.
7.	$7 + 4 = 4 + 7$	7. Commutative postulate.

The chain of statements shows that the following sentence is true.

$$4 - (-7) = 4 + (+7)$$
To subtract -7, add $+7$.

The proof for any real numbers, a and b, follows the same pattern as in the example. The proof of the general theorem is left as an exercise. It is stated in words and mathematical symbols as follows.

Theorem 2-3 To subtract any real number b from any real number a, add the additive inverse of b to a.

$$\forall a \in R \quad \forall b \in R \quad a - b = a + (-b)$$

Exercises

A — Name each difference with a single numeral.

1. $5 - 11$	**2.** $-8 - (-6)$	**3.** $-4 - 5$
4. $-5 - 1$	**5.** $-6 - 2$	**6.** $4 - (-6)$
7. $-6 - 4$	**8.** $-6 - (-4)$	**9.** $-4 - 6$
10. $-8 - 3$	**11.** $-4 - 8$	**12.** $-6 - (-7)$
13. $16 - (-15)$	**14.** $-24 - 12$	**15.** $15 - 15$
16. $-9 - 3$	**17.** $-9 - 10$	**18.** $-10 - 10$
19. $-2.4 - 9.4$	**20.** $0 - (-7.8)$	**21.** $-7.8 - 0$
22. $\frac{1}{2} - (-\frac{1}{2})$	**23.** $\frac{5}{3} - (-\frac{2}{3})$	**24.** $-54 - (-14)$
25. $-64 - 74$	**26.** $12.3 - 6.7$	**27.** $-12.3 - 6.7$
28. $12.3 - (-6.7)$	**29.** $6.7 - 12.3$	**30.** $-12.3 - (-6.7)$
31. $-6.7 - 12.3$	**32.** $-6.7 - (-12.3)$	**33.** $6.7 - (-12.3)$
34. $-\frac{1}{2} - (-\frac{1}{2})$	**35.** $-\sqrt{2} - \sqrt{2}$	**36.** $\sqrt{3} - (-\sqrt{3})$

— Label each of the following as *True* or *False*. The replacement set for a and b is R.

37. $-6 + (-2) = 4$	**38.** $-\frac{2}{3} - (-\frac{2}{3}) = \frac{1}{3}$
39. $-4 - (-2) = -2 - (-4)$	**40.** $\lvert 0 \rvert = 0.24 - (-0.24)$
41. $8 - (-8) = 0$	**42.** $6 - 2 = 6 + (-2)$
43. $\lvert -8 - (-2) \rvert = \lvert -8 \rvert - \lvert -2 \rvert$	**44.** $\lvert 8 - 2 \rvert = \lvert 2 - 8 \rvert$
45. $\lvert 8 - (-2) \rvert = \lvert 8 \rvert - \lvert -2 \rvert$	**46.** $\lvert 8 - 2 \rvert = \lvert 8 \rvert - \lvert 2 \rvert$
47. $\lvert 2 - 8 \rvert = \lvert 2 \rvert - \lvert 8 \rvert$	**48.** $-3 + (-3) = -6$
49. $a - b = a + (-b)$	**50.** $\lvert a - b \rvert = \lvert b - a \rvert$

B — The formula for converting a temperature expressed in degrees Fahrenheit to a temperature expressed in degrees Celsius (centigrade) is $C = \frac{5}{9}(F - 32)$, where C represents the number of degrees Celsius and F represents the number of degrees Fahrenheit. For each of the following temperatures expressed in degrees Fahrenheit, find the temperature in degrees Celsius.

51. $70°$ **52.** $32°$ **53.** $50°$ **54.** $100°$

C **55.** Use the method in the Example on page 73 to prove that $a - b = a + (-b)$ is true for all real replacements for a and b.

56. Prove that $\lvert a - b \rvert = \lvert b - a \rvert$ is true for all real replacements for a and b. (*Hint:* You will have to consider several cases.)

2–8 Multiplication of Real Numbers

You know that there are certain postulates that are true for multiplication of *positive* numbers. Since the set of positive numbers is a subset of the set of real numbers, it is reasonable to expect that those postulates also apply to the set of real numbers.

Recall that xy means the same as $x \cdot y$, $x(y)$, and $x \times y$.

Postulates for Multiplication of Real Numbers		Name
2–7	$\forall x \in R \quad \forall y \in R \quad xy$ **is one and only one real number.**	Closure postulate.
2–8	$\forall x \in R \quad \forall y \in R \quad xy = yx$	Commutative postulate.
2–9	$\forall x \in R \quad \forall y \in R \quad \forall z \in R \quad (xy)z = x(yz)$	Associative postulate.
2–10	$\forall x \in R \quad x \cdot 1 = 1 \cdot x = x$, **where 1 is the only number with this property.**	Identity postulate.
2–11	$\forall x \in R, x \neq 0$ **there exists** $\frac{1}{x}$ **such that** $x \cdot \frac{1}{x} = \frac{1}{x} \cdot x = 1$, **where for each** x **there is only one number** $\frac{1}{x}$.	Inverse postulate.
2–12	$\forall x \in R \quad \forall y \in R \quad \forall z \in R \quad x(y + z) = xy + xz$	Distributive postulate for multiplication over addition.

You know how to find the product of two positive real numbers. How do you find the product of two real numbers when one is positive and one is negative? The example below shows how to find that product, but first you need the following theorem.

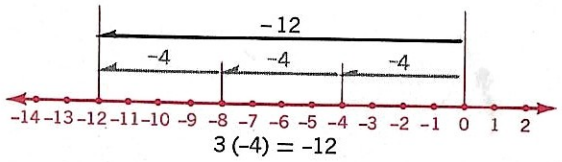
Theorem 2–4 The product of any real number and 0 is 0.

$$\forall x \in R \quad x \cdot 0 = 0 \cdot x = 0$$

The proof of the theorem is left as an exercise.

EXAMPLE. Show that $3(-4) = -12$.

One way to do this is to use a vector diagram on a number line.



The diagram illustrates the process of multiplication as repeated addition. You can think of -4 as being a "loss of 4" and of 3 as being "3 times." Then, 3 losses of 4 each is a total loss of 12, or -12.

You may also be interested in the proof that $3(-4) = -12$. The postulates and previously proved theorems are used as reasons.

The goal in proving the theorem is to show that $3 \cdot 4$ and $3(-4)$ have a sum of 0. If the sum is 0, then $3 \cdot 4$ and $3(-4)$ are additive inverses, and $3(-4)$ is -12, the inverse of 12.

You begin the proof of the theorem by choosing the sentence $4 + (-4) = 0$. This will lead to the desired result. (Notice that addition and subtraction will be indicated in black type from now on. You can differentiate between the signs by the way the sign is used.)

	Statements	*Reasons*
1.	$4 + (-4) = 0$	1. Additive inverse postulate.
2.	$3[4 + (-4)] = 3 \cdot 0$	2. Closure postulate.
3.	$3[4 + (-4)] = 0$	3. Theorem 2–4.
4.	$3 \cdot 4 + 3(-4) = 0$	4. Distributive postulate.
5.	Hence, $3 \cdot 4$ and $3(-4)$ are additive inverses.	5. Additive inverse postulate.
6.	$3(-4) = -(3 \cdot 4)$, or -12.	6. Additive inverse postulate.

Having proved that $3(-4) = -12$ is a true sentence, you can use the commutative postulate to show that $(-4) \cdot 3$ is also equal to -12.

Use vector diagrams to show that each of the following is true. Try to make a rule for the product of a positive and a negative number.

a. $5(-4) = -20$ **b.** $8(-3) = -24$ **c.** $7(-2\frac{1}{2}) = -17\frac{1}{2}$
d. $(-5)(8) = -40$ **e.** $(-6)(5) = -30$ **f.** $(-9)(9) = -81$

The generalization for the product of a positive number and a negative number is stated as a theorem. The proof is left as an exercise.

Theorem 2–5 The product of a and $-b$ is $-ab$.

$$\forall a \in R \quad \forall b \in R \quad a(-b) = -(ab)$$

The proof of Theorem 2–6 follows directly from Theorem 2–5.

Theorem 2–6 The product of $-a$ and b is $-ab$.

$$\forall a \in R \quad \forall b \in R \quad -a(b) = -(ab)$$

	Statements	*Reasons*
1.	$-a(b) = b(-a)$	1. Commutative postulate.
2.	$= -(ba)$	2. Theorem 2–5.
3.	$= -(ab)$	3. Commutative postulate.

Now try these

Find the products.

1. $7(-3)$ 2. $-6 \cdot 9$ 3. $4 \times (-8)$ 4. $(-5)(0)$
5. $3.2(-2.1)$ 6. $-0.5(4.6)$ 7. $\frac{3}{4}(-\frac{2}{3})$ 8. $(-15)(8)$

Answers: 1. -21 2. -54 3. -32 4. 0 5. -6.72 6. -2.3 7. $-\frac{1}{2}$ 8. -120

Exercises

A Find the products of the following numbers.

1. 6 and -3 2. -5 and 2 3. 2 and -5
4. -2 and 5 5. 5 and -2 6. 2 and 5
7. 4 and 0 8. -4 and 0 9. -4 and 3
10. 4 and -3 11. $-\frac{3}{4}$ and $\frac{3}{8}$ 12. $\frac{1}{8}$ and $-\frac{1}{8}$

Find the products as indicated.

13. $6 \cdot 3$	**14.** $-6 \cdot 3$	**15.** $-3 \cdot 6$
16. $8(-\frac{1}{4})$	**17.** $12(6)$	**18.** $12(-6)$
19. $-12(6)$	**20.** $(-1)(5)$	**21.** $8(-2.5)$
22. $\frac{2}{3}(-12)$	**23.** $3.8(-10)$	**24.** $-62(100)$
25. $-17(28)$	**26.** $2\frac{1}{5}(-\frac{3}{5})$	**27.** $33 \cdot (-\frac{1}{3})$
28. $8.1(0)$	**29.** $-5(2.4)$	**30.** $-7.2(\frac{1}{4})$

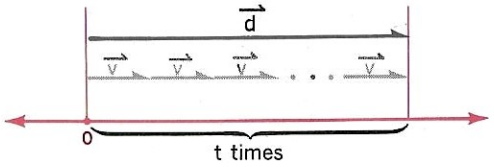
B Recall that a Celsius (centigrade) temperature C may be found from a Fahrenheit temperature F by the formula $C = \frac{5}{9}[F + (-32)]$. Give the Celsius temperature that corresponds to each of the following Fahrenheit temperatures.

31. 7° **32.** 43° **33.** −18° **34.** 212° **35.** −54°

Using the formula $\frac{9}{5}C + 32 = F$, give the Fahrenheit temperature that corresponds to each of the following Celsius temperatures.

36. 72° **37.** 20° **38.** 0° **39.** −12° **40.** −2°

41. You are familiar with the statement that distance is the product of rate and time, or $d = rt$. When you consider direction along with distance, the formula becomes a vector formula, $\vec{d} = \vec{v}t$.



$$\vec{d} = \vec{v} \times t$$

distance in given direction = velocity in given direction × time

The time is always a positive number; it has no direction.

If traveling east or north is viewed as moving in the positive direction and traveling west or south is viewed as moving in the negative direction, find d if

a. you travel south at 10 miles per hour for $6\frac{1}{2}$ hours.

b. you travel north at 50.2 kilometers per hour for 5 hours.

c. you walk east at 200 meters per minute for 8 minutes and west at 180 meters per minute for 6 minutes.

C **42.** Show that the two formulas for converting temperatures, $\frac{9}{5}C + 32 = F$ and $C = \frac{5}{9}[F + (-32)]$, are equivalent.

For the following proofs, copy the *Statements* column as shown, and then give the reason for each directly to the right.

43. Prove Theorem 2–4: $\forall x \in R \quad x \cdot 0 = 0 \cdot x = 0$.

Statements	*Reasons*
1. a. $x(1 + 0) = x \cdot 1 + x \cdot 0$	**1. a.** ?
b. $= x + x \cdot 0$	**b.** ?
2. a. $x(1 + 0) = x(1)$	**2. a.** ?
b. $= x$	**b.** ?
3. Therefore, $x = x + x \cdot 0$.	**3.** ?
4. Hence, $x \cdot 0 = 0$.	**4.** ?
5. And $0 \cdot x = 0$.	**5.** ?

44. Prove Theorem 2–5: $\forall a \in R \quad \forall b \in R \quad a(-b) = -(ab)$.

Statements	*Reasons*
1. $b + (-b) = 0$	**1.** ?
2. $a[b + (-b)] = a(0)$	**2.** ?
3. $ab + a(-b) = 0$	**3.** ?
4. Therefore, $a(-b) = -(ab)$.	**4.** ?

2–9 The Product of Two Negative Numbers

To find the replacement for n that makes $-3(-4) = n$ a true sentence, you must be able to find the product of two negative numbers.

Look at the table of products shown at the right. Is the first factor in each sentence the same? What pattern do you see for the second factor in the sentences? What pattern do you see for the products?

$-3(4) = -12$
$-3(3) = -9$
$-3(2) = -6$
$-3(1) = -3$
$-3(0) = 0$
$-3(-1) = ?$
$-3(-2) = ?$
$-3(-3) = ?$
$-3(-4) = ?$

If the pattern for the products continues, what will the last four products be? Does it make sense for $-3(-1)$ to be 3? for $-3(-2)$ to be 6? for $-3(-3)$ to be 9? for $-3(-4)$ to be 12?

Here is another way to reason about the problem. Think of −4 as a "loss of $4 a week"

and of -3 as "3 weeks ago." If you have lost \$4 a week for the past 3 weeks $[(-4) \times (-3)]$, then 3 weeks ago you were \$12 richer $(+12)$.

You may be interested in the way to prove that $-3(-4) = 12$ is true by using the postulates and previous theorems.

	Statements	*Reasons*
1.	$4 + (-4) = 0$	1. Additive inverse postulate.
2.	$-3[4 + (-4)] = -3(0)$	2. Closure postulate.
3.	$-3(4) + (-3)(-4) = 0$	3. Distributive postulate and Theorem 2–4.
4.	$-12 + (-3)(-4) = 0$	4. Theorem 2–6.

Since -12 and $(-3)(-4)$ have a sum of 0, they are additive inverses. By the additive inverse postulate, a number has only one additive inverse. Since 12 is the additive inverse of -12, then $(-3)(-4)$ is equal to 12.

Is each of the following sentences true? Explain each one by using any of the ways of thinking in this section. As you work, try to make a rule for multiplying two negative numbers.

a. $-7(-5) = 35$ **b.** $-8(-2) = 16$ **c.** $-7\frac{1}{2}(-2) = 15$

d. $-\sqrt{2}(-\sqrt{2}) = 2$ **e.** $-\sqrt{3}(-\sqrt{3}) = 3$ **f.** $-5(-15) = 75$

g. $-0.5(-4) = 2$ **h.** $-0.3(-0.7) = 0.21$ **i.** $-\frac{3}{4}(-\frac{2}{3}) = \frac{1}{2}$

If $-a$ is negative and $-b$ is negative, is their product positive or negative?

The conclusion you might draw from **a–i** can be summarized as a theorem.

Theorem 2–7 The product of $-a$ and $-b$ is ab.

$$\forall a \in R \quad \forall b \in R \quad (-a)(-b) = ab$$

The proof is left as an exercise.

Exercises

A — Find the products of the following numbers.

1. -2 and -5 **2.** -5 and -2 **3.** -3 and 5

4. -4 and 3 **5.** -4 and -3 **6.** 4 and -3

7. -2.5 and -0.1 **8.** $-\frac{1}{8}$ and $-\frac{5}{6}$ **9.** $-\sqrt{5}$ and $-\sqrt{5}$

— Find the products as indicated.

10. $-6(-3)$ **11.** $-4(-5)$ **12.** $4(-5)$

13. $-4(5)$ **14.** $4(5)$ **15.** $0(-2)$

16. $-2(-9)$ **17.** $7(-1)$ **18.** $-48(-10)$

19. $-1(-1)$ **20.** $-\frac{2}{3}(-\frac{3}{2})$ **21.** $-7649(-580)$

22. $-0.5(-12)$ **23.** $-1.1(3.4)$ **24.** $-\frac{1}{9}(-\frac{3}{4})$

25. What postulate for the real numbers can you use in finding the product of three numbers?

26. Do you think the following is true?

$$[-2(4)](-3) = -2[4(-3)]$$

Why or why not? Check by multiplying both ways.

— Find the products as indicated.

27. $5(4)(-2)$ **28.** $-5(4)(-2)$ **29.** $-6(-5)(-4)$

30. $5(-4)(-2)$ **31.** $6(-4)(-5)$ **32.** $-8(-1)(2)$

33. $3(0)(-5)$ **34.** $5(-6)(-4)$ **35.** $-2(-2)(-2)$

36. $(-3)^2 = (-3)(-3)$ **37.** $(-4)^2$ **38.** $(5)^2$

39. $(-1)^2 = (-1)(-1)$ **40.** $(1)^2$ **41.** $(-1)^3$

42. What kind of number is the product when two negative numbers are multiplied? when four negative numbers are multiplied?

43. What kind of number is the product when three negative numbers are multiplied? five negative numbers?

44. For the product to be positive, must an even number of negative numbers be multiplied or an odd number of negative numbers? Explain.

45. Is the set of negative real numbers *closed* under multiplication?

— Replace ● with either $>$, $<$, or $=$ to make true statements.

46. $2(-3)$ ● $-3(2)$ **47.** $-6(-1)$ ● $-1(6)$

48. $-0.34(-6)$ ● $-0.17(-3)$ **49.** $-\frac{a}{b}(\frac{1}{a})$ ● $-\frac{1}{b}(\frac{a}{a})$

50. $-6(0)$ ● $0(6)$ **51.** $|4 \cdot 5|$ ● $|4| \cdot |5|$

52. $|-0.34 \cdot -2|$ ● $|-0.34| \cdot |-2|$ **53.** $|0 \cdot 1|$ ● $|0| \cdot |1|$

54. $|-0.2 \cdot 0.1|$ ● $|-0.2| \cdot |0.1|$ **55.** $|\frac{a}{b} \cdot \frac{1}{a}|$ ● $|\frac{a}{b}| \cdot |\frac{1}{a}|$

B — Substitute as indicated: $r = -3$, $s = 2$, $t = -7$. Express your answers as real numbers.

56. rs **57.** tr **58.** sr **59.** rst

60. $2st$ **61.** $3rs$ **62.** $\frac{1}{3}rt$ **63.** $(-4)(st)$

64. $-\frac{1}{2}(rs)$ **65.** $-2(rs)$ **66.** $\frac{1}{6}rst$ **67.** $-3rst$
68. $s(r+t)$ **69.** $r(s+2t)$ **70.** $s(t+r)$ **71.** $t(s+r)$
72. $r(s+t)$ **73.** $s(r-t)$ **74.** $s(r-2t)$ **75.** $-s(r-3t)$

76. Complete the following proof of Theorem 2–7: The product of the opposites of two numbers is equal to the product of the two numbers. Copy the statements and supply a reason for each.

	Statements	*Reasons*
1.	$b+(-b)=0$	1. ?
2.	$-a[b+(-b)]=-a(0)$	2. ?
3.	$-a(b)+(-a)(-b)=0$	3. ?
4.	$-ab+(-a)(-b)=0$	4. ?
5.	$(-a)(-b)=ab$	5. ?

2–10 Division of Real Numbers

Look at the pairs of sentences and solutions, and see if you can explain each.

Find c.	Think	Quotient
a. $\frac{15}{5}=c$	$5\cdot c=15$	3
b. $\frac{3}{9}=c$	$9\cdot c=3$	$\frac{1}{3}$
c. $\frac{10}{-5}=c$	$(-5)\cdot c=10$	-2
d. $\frac{-20}{-4}=c$	$(-4)\cdot c=-20$	5

To find the quotient for $\frac{15}{5}$, you can ask, "5 times what number gives 15?" Thus, as subtraction is the inverse of addition, *division is the inverse of multiplication.* By using multiplication to define division, $\frac{15}{5}$ means to find a number such that 5 times that number is 15; that is, $\frac{15}{5}$ means $5\cdot c=15$.

Definition To divide a by b $(b\neq 0)$ find the number c such that $b\cdot c=a$ is true.

$\forall a\in R \quad \forall b\in R\ (b\neq 0) \quad \forall c\in R \quad \frac{a}{b}=c$ **if and only if** $b\cdot c=a$

Another way to solve the division problem is to multiply the dividend by the reciprocal of the divisor.

EXAMPLE 1. $\frac{24}{6}=\underline{?}$
Think: $\frac{24}{6}=24\times\frac{1}{6}=\underline{?}$. The quotient is 4.
Thus, $\frac{24}{6}=4$ is true.

EXAMPLE 2. $\frac{-24}{-6}=\underline{?}$
Think: $-24\cdot(-\frac{1}{6})=\underline{?}$. The quotient is 4.
Thus, $\frac{-24}{-6}=4$ is true.

EXAMPLE 3. $\frac{24}{-6}=\underline{?}$
Think: $24\cdot(-\frac{1}{6})=\underline{?}$. The quotient is -4.
Thus, $\frac{24}{-6}=-4$ is true.

EXAMPLE 4. $\frac{-24}{6}=\underline{?}$
Think: $-24\cdot\frac{1}{6}=\underline{?}$. The quotient is -4.
Thus, $\frac{-24}{6}=-4$ is true.

Whichever method of thinking you use, it should be clear from the examples above that the pattern for positive and negative numbers in division is the same as the pattern in multiplication. The quotient of two positive numbers is positive. The quotient of two negative numbers is positive. The quotient of a negative number and a positive number is negative.

$$a\div b=a\div b \qquad -a\div b=-(a\div b)$$
$$-a\div -b=a\div b \qquad a\div -b=-(a\div b)$$

Exercises

A — Divide as indicated.

1. $\frac{-6}{2}$ **2.** $\frac{-6}{-2}$ **3.** $\frac{6}{-2}$ **4.** $\frac{6}{2}$
5. $\frac{-20}{5}$ **6.** $\frac{-20}{-5}$ **7.** $\frac{20}{5}$ **8.** $\frac{20}{-5}$
9. $\frac{0}{5}$ **10.** $\frac{0}{-5}$ **11.** $\frac{100}{-2}$ **12.** $\frac{-600}{5}$

— Divide each of the following numbers by -5.

13. 10 14. -5 15. 0 16. -20
17. 20 18. -15 19. $-5x$ 20. $5y$

— The definition of division includes the condition that you cannot divide by 0. By what number can x not be replaced in each of the following? (Ask yourself what replacement for x will make the denominator 0?)

21. $\frac{6}{x}$ 22. $\frac{8}{x-1}$ 23. $\frac{10}{-5+x}$
24. $\frac{5}{2x-4}$ 25. $\frac{7}{-1+3x}$ 26. $\frac{8}{2x+12}$

— Divide as indicated.

27. $\frac{7}{10} \div 14$ 28. $-\frac{3}{7} \div (-\frac{8}{21})$ 29. $-13\frac{1}{2} \div 3\frac{5}{6}$
30. $-3241 \div (-10)$ 31. $785 \div (-100)$ 32. $3.2 \div 1000$
33. $15.75 \div (-15)$ 34. $-4590 \div 45$ 35. $-6\frac{2}{3} \div 3\frac{3}{4}$

— Replace the letters by real numbers to make true sentences.

36. $\frac{-100}{-75} = a$ 37. $\frac{c}{-8} = 12$ 38. $\frac{\frac{1}{15}}{-\frac{1}{14}} = a$
39. $\frac{-\frac{1}{4}}{\frac{3}{8}} = b$ 40. $\frac{-14}{\frac{2}{7}} = a$ 41. $-\frac{-7}{c} = -21$
42. $-\frac{9}{b} = 18$ 43. $\frac{a}{3} = 0$ 44. $\frac{c}{2} = -6$

B — Perform the indicated operations.

45. $(3)(-5) + (\frac{-8}{2})$ 46. $(-3)(-4)[-(\frac{-9}{-3})]$
47. $(7)^2 + 2(7)^2(-3)$ 48. $(4)^3 - 5(3)(-2)$

— Find the numbers that result if the replacement set for x in each of the following expressions is $\{-3, -1, 0, 2\}$.

49. $x + 5$ 50. $x - 5$ 51. $2x$
52. $\frac{x-6}{2}$ 53. $x^2 + 5$ 54. $x^2 + x$

— Substitute as indicated and perform the operations.

$r = 4$ $s = 6$ $t = (-3)$
$u = (-2)$ $v = (-3)$ $w = (-\frac{1}{2})$

55. $s + t$ 56. $st + rw$ 57. $tu + rs$
58. $uv + rs$ 59. $s \div u$ 60. $r \div u$
61. $r \div w$ 62. $s \div t$ 63. $sw \div t$
64. $st - r$ 65. $rw \div u$ 66. $3(r \div w)$

67. $\frac{u}{v} - \frac{s}{r}$ 68. $stu \div v$ 69. $(r \div w) + t$
70. $(sv \div w) - r$ 71. $(svw \div t) + u$ 72. $\frac{s}{u} \div \frac{v}{t}$

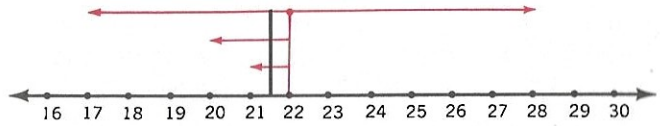
C — You know how to find the mean, or average, of a group of numbers by adding the numbers and then dividing the sum by the number of addends. But for a long list of large numbers, it is possible to find the mean more easily by using an *estimation*.

Example. Find the mean of 17, 20, 28, and 21.

Try 22 as the mean. The difference between each number and the mean is called the **deviation from the mean,** and the sum of the deviations from the *true* mean is always zero. If 22 is the true mean, the sum of the deviations is zero.

$$\begin{aligned} 17 - 22 &= -5 \\ 20 - 22 &= -2 \\ 28 - 22 &= 6 \\ 21 - 22 &= -1 \\ &\overline{-2} \end{aligned}$$

Since the sum of the deviation is -2, not zero, 22 is not the true mean. Can you find the true mean from the assumed mean?



There are four deviations and each one is greater than or less than the deviation from the true mean by the same amount. In this example, the sum is -2, so each deviation is $\frac{-2}{4}$, or $-\frac{1}{2}$, unit different from the deviation from the true mean; that is, each is $\frac{1}{2}$ unit less than it should be. Then the true mean is $-\frac{1}{2}$ unit away from the assumed mean. So the true mean is $22 + (-\frac{1}{2})$, or $21\frac{1}{2}$. Check to see that the sum of the deviations from $21\frac{1}{2}$ is zero.

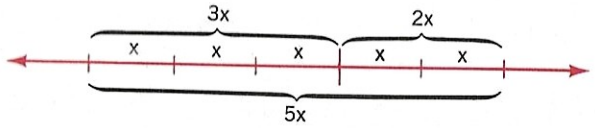
73. Use an assumed mean to find the true mean of 29, 42, 50, 37, 45, 30, 30, and 49.

74. Using 152 as an assumed mean, find the mean weight of 16 boys, whose weights are given below.

116	200	184	150
110	195	140	136
162	171	158	147
130	125	118	167

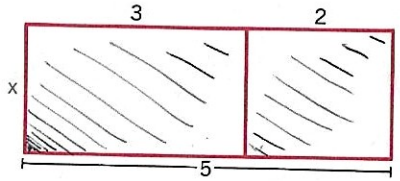
2–11 Equivalent Algebraic Expressions

Is it reasonable to suspect that the expressions $3x + 2x$ and $5x$ name the same number for every real number x? Use a line segment as a model for x.



It would seem that $3x + 2x$ and $5x$ would always be equivalent.

Use the area of a rectangle as a model. What is the area of each of the smaller rectangles? What is the area of the large rectangle?



Then, $3x + 2x$ and $5x$ are again names for the same thing.

Experiment further by choosing a few real-number replacements for the variable in an expression and then compare the numbers that result.

EXAMPLE 1. Are $3x + 2x$ and $5x$ equivalent?

Replacements for x	$3x + 2x$	$5x$
2	$3(2) + 2(2) = \underline{?}$ $6 + 4 = 10$	$5(2) = 10$
-3	$3(-3) + 2(-3) = \underline{?}$ $(-9) + (-6) = -15$	$5(-3) = -15$
$\frac{1}{2}$	$3(\frac{1}{2}) + 2(\frac{1}{2}) = \underline{?}$ $1\frac{1}{2} + 1 = 2\frac{1}{2}$	$5(\frac{1}{2}) = 2\frac{1}{2}$
-0.7	$3(-0.7) + 2(-0.7) = \underline{?}$ $(-2.1) + (-1.4) = -3.5$	$5(-0.7) = -3.5$

For each of these replacements for x, $3x + 2x$ and $5x$ name the same number. Do you think there are other values of x for which $3x + 2x$ and $5x$ name the same number?

Definition Two algebraic expressions are equivalent expressions if and only if the expressions name the same number for every replacement of the variables for which the expressions are defined.

If you find that each of several randomly chosen replacements gives the same numbers for the two algebraic expressions, you may reasonably expect that the algebraic expressions are equivalent. Thus, in the above illustration, you may reasonably conclude that $3x + 2x$ and $5x$ are equivalent and write $\forall x\ 3x + 2x = 5x$.

EXAMPLE 2. Are $x(x + 1)$ and $2x$ equivalent?

Using the same reasoning as in Example 1, you obtain the following.

Replacements for x	$x(x + 1)$	$2x$
0	$0(0 + 1) =$ $0 \cdot 1 = 0$	$2(0) = 0$
-1	$-1(-1 + 1) =$ $-1(0) = 0$	$2(-1) = -2$
$\frac{2}{3}$	$\frac{2}{3}(\frac{2}{3} + 1) =$ $\frac{2}{3} \cdot \frac{5}{3} = \frac{10}{9}$	$2(\frac{2}{3}) = \frac{4}{3}$
-1.2	$(-1.2)(-1.2 + 1) =$ $(-1.2)(-0.2) = 0.24$	$2(-1.2) = -2.4$

You could have stopped when x was replaced by -1. Since you found a real number for which $x(x + 1)$ and $2x$ do not name the same number, you can conclude that $x(x + 1)$ and $2x$ are *not equivalent*.

Exercises

A Each of the following exercises contains two algebraic expressions that may or may not be equivalent. Following the form in Examples 1 and 2, use four replacements for the variable and then state whether or not you can reasonably conclude that the expressions are equivalent.

1. $3x + 5x$ and $8x$
2. $x(x + 2)$ and $3x$
3. x and $2x$
4. $6x - x$ and $5x$
5. $2(x + y)$ and $2x + 2y$
6. $-7m + (-6m)$ and $-13m$
7. $3x + 3y$ and $3(x + y)$
8. $x + x + x$ and $3x$
9. $|x + y|$ and $|x| + |y|$

B **10.** $|x \cdot y|$ and $|x| \cdot |y|$ **11.** $\frac{|x|}{y}$ and $\frac{|x|}{|y|}$

12. $|x - y|$ and $|x| - |y|$ **13.** $\frac{(a+b)(a-b)}{a-b}$ and $a+b$

14. $a^2 - b^2$ and $(a-b)^2$ **15.** $(a+b)^2$ and a^2+b^2

16. $a(b+cd)$ and $ab+ac-ad$ **17.** x^2y and $x(xy)$

18. $a-b$ and $a+(-b)$ **19.** $\frac{a+b}{b}$ and $a+1$

C — Find the solution set for these sentences if the replacement set is the set of integers.

20. $3x+5x=8x$ **21.** $3x=x+x+x$

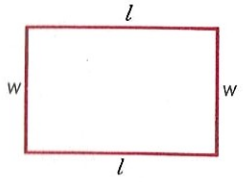
22. $x+0=x+1$ **23.** $6x-x=5x$

24. $x-2\frac{1}{5}=5\frac{4}{5}$ **25.** $3x=-15$

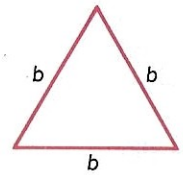
2–12 Simplifying Algebraic Expressions

In the preceding section, did you notice that of two equivalent expressions, one may be simpler in appearance than the other? The expressions $3(t^2-6n)-5t^2+15n+2t(t+\frac{5}{2})+5n$ and $5t+2n$ are equivalent, but the second is much simpler than the first. The process of obtaining an equivalent expression that is simpler in appearance than a given expression is called **simplifying an algebraic expression.**

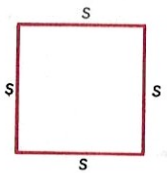
The formulas you use for the perimeters of polygons are expressions that have been simplified.



$P = w+l+w+l$
$= 2w+2l$



$P = b+b+b$
$= 3b$



$P = s+s+s+s$
$= 4s$

To simplify algebraic expressions, use the postulates for the real numbers, usually the distributive postulate. Study the following.

a. $3a+5a+7b = (3+5)a+7b$
$= 8a+7b$

b. $3x^2y+5x^2y+7x = (3+5)x^2y+7x$
$= 8x^2y+7x$

c. $\frac{1}{2}ab^3+\frac{1}{3}ab^3+\frac{2}{5}a^2b = (\frac{1}{2}+\frac{1}{3})ab^3+\frac{2}{5}a^2b$
$= \frac{5}{6}ab^3+\frac{2}{5}a^2b$

d. $\frac{1}{2}mn+\frac{2}{5}m+\frac{1}{3}mn = \frac{1}{2}mn+\frac{1}{3}mn+\frac{2}{5}m$
$= (\frac{1}{2}+\frac{1}{3})mn+\frac{2}{5}m$
$= \frac{5}{6}mn+\frac{2}{5}m$

Notice that the distributive postulate is used only for terms that have the same variable or variables as factors. Thus, $3a+5a+7b$ can be simplified by using the distributive postulate on $3a+5a$ because each of these terms has the same factor, a. And notice that the commutative postulate was used in **d** to transfer $\frac{2}{5}m$ and $\frac{1}{3}mn$ so the distributive postulate could be used.

If you know the numerical values of the variables in each of the simplified expressions in **a–d** above, the resulting expressions can be written as one numeral. For example, if a is 3 and b is 5, you can write $8a+7b$ as $8(3)+7(5)$, or $24+35$, or 59.

Study these examples to become more familiar with simplifying algebraic expressions.

EXAMPLE 1. Simplify $a+3a$.

By the multiplicative identity postulate, $\forall a \;\; 1 \cdot a = a \cdot 1 = a$.

Thus, to simplify $a+3a$, you use the identity postulate and the distributive postulate.

$$a+3a = 1a+3a = (1+3)a = 4a$$

EXAMPLE 2. Simplify $8x-3x$.

To use the distributive postulate, think $8x-3x$ is $8x+(-3x)$.

$$8x-3x = 8x+(-3x) = [8+(-3)]x = 5x$$

EXAMPLE 3. Combine $-a$ and $3a$.

To simplify this expression, you use the fact that $-a = (-1)a$ is true and the distributive postulate.

$$-a+3a = (-1)a+3a = (-1+3)a = 2a$$

EXAMPLE 4. Combine $6ab$ and $7ba$.

Terms such as $6ab$ and $7ba$ have the same factors and the order of the factors in each term may be interchanged by applying the commutative postulate of multiplication: $ba = ab$.

$6ab+7ba = 6ab+7ab$ Commutative postulate.
$= (6+7)ab$ Distributive postulate.
$= 13ab$ Closure postulate.

EXAMPLE 5. Simplify $3(a + 2) + 2a - 6$.

$$
\begin{aligned}
3(a+2)+2a-6 &= (3a+6)+2a-6 && \text{Distributive postulate.}\\
&= 3a+(6+2a)-6 && \text{Associative postulate.}\\
&= 3a+(2a+6)-6 && \text{Commutative postulate.}\\
&= (3a+2a)+(6-6) && \text{Associative postulate.}\\
&= 5a+0 && \text{Closure and additive inverse postulates.}\\
&= 5a && \text{Additive identity postulate.}
\end{aligned}
$$

You know that a simpler expression for $3a + 7b + 6a$ is $9a + 7b$. You can say that $3a + 7b + 6a = 9a + 7b$ is true for all replacements of a and b. When you simplify an algebraic expression, the original expression and the simplified expression will always have the same value when the variables are replaced by numbers.

EXAMPLE 6. Replace a by 8 and b by 4 to show that $3a + 7b + 6a = 9a + 7b$ is true.

$$
\begin{aligned}
3a+7b+6a &= 3\cdot 8+7\cdot 4+6\cdot 8\\
&= 24+28+48\\
&= 100
\end{aligned}
$$

$$
\begin{aligned}
9a+7b &= 9\cdot 8+7\cdot 4\\
&= 72+28\\
&= 100
\end{aligned}
$$

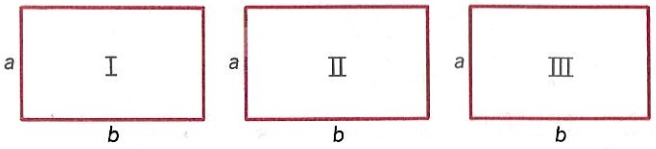
You can see that both expressions name the same number.

Exercises

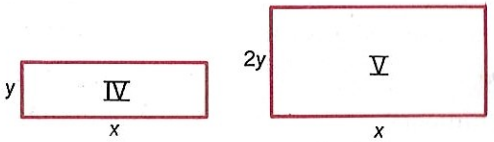
A — Simplify each of the following.

1. $3x + 5x$
2. $7m + 6m$
3. $3a + 7b + 2a$
4. $6ab + 7ab$
5. $7x + 2x + 3y + 5y$
6. $8xy + 3xy + x + 3x$
7. $10mn + 3mn + 6y$
8. $12x + 3y + x + 2y$
9. $4ab + 7ba$
10. $3(a + b) + 4(a + b)$
11. $5xy + 7xy + 8x + x$
12. $2\frac{1}{2}mn - mn + 3\frac{1}{3}nm$
13. $10r - 2r + 3r + 5r - 6s + 6s$
14. $15xy - 2xy - 7y + 4$
15. $10x + 3r - r + x + 3x$
16. $19m + m - 3n + 7n + 2n$
17. $4xy + xy + 3yx + 4yx$
18. $7(x - y) + 6(x - y)$
19. $4\frac{1}{2}(x-y)+7\frac{1}{3}(x-y)+2\frac{1}{4}(x-y)$
20. $42xy + 75xy + 7x^2y + 8xy^2$

B **21.** The area of rectangular region I is ab. The area of region II is ?. The area of region III is ?. Find the combined area of regions I, II, and III.



The area of region IV is ?; of region V is ?.



The combined area of all five rectangular regions is ?.

If x is replaced by 9 and y by 6, does the expression in Column I have the same value as the expression in Column II? Answer *Yes* or *No*. Be prepared to give a reason.

	Column I	Column II
22.	$3x + 4y - 5x + 2y$	$8x + 6y$
23.	$3x^2 + 7xy - x^2y$	$2x^2y + 7xy^2$
24.	$4x^2y + 5x^2y + y^2$	$9x^2y^3$
25.	$3xy + 7yx + 2x^2$	$10x^2y^2 + 2x^2$
26.	$12x + 3y + 2x + y$	$14x + 4$
27.	$3(x - y) + 7(x - y)$	$10(x - y)$
28.	$8(x + y) + 3(x + y)$	$11(x + y)$
29.	$2xy + 3y + 7yx + 2y$	$9xy + 6y$
30.	$2x + 5x + x + 7y + 4$	$7x + 7y + 4$
31.	$4xy + 9yx + 2xy$	$15\frac{1}{2}xy$

32. Are the expressions in Exercises 22–31 equivalent if x and y can be replaced by any numbers?

C — Write the solution set for each of the following. The replacement set is the set of real numbers.

33. $3a + 10a = 13a$
34. $3x - 4 = 8$
35. $\frac{x}{3} - 1 = 7$
36. $\frac{3x}{2} + 4 = 10$
37. $5x + 2(3x + 7) = 59$
38. $\frac{1}{3}(4x + 1) = 5 + x$

Write each of the following in its simplest form.

39. $3 \cdot 5a + 7 \cdot 3b + 3 \cdot 2a + 2 \cdot 9b$

40. $\frac{5}{8}c + \frac{1}{4}d + \frac{3}{7}c + \frac{3}{5}d$

41. $7x + 3y + 2z + x + 4y + 6z$

42. $2c + 1\frac{1}{2}d + 1\frac{1}{4}c + 2\frac{3}{4}d$

43. $5x - y + 3 \cdot 5z - 1\frac{1}{6}z + 2\frac{1}{2}x + 3\frac{1}{4}y$

CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

integers (*52*)
rational numbers (*53*)
irrational numbers (*53*)
real numbers (*53*)
unit square (*53*)
unit circle (*54*)
opposites (*56*)
negative of (*56*)
absolute value (*57*)
vector (*59*)
closure postulates for real numbers (*63, 75*)
commutative postulates for real numbers (*63, 75*)
associative postulates for real numbers (*63, 75*)
identity postulates (*63, 75*)
inverse postulates (*64, 75*)
theorem (*67*)
subtraction of real numbers (*72*)
distributive postulate (*75*)
division of real numbers (*82*)
deviation from the mean (*85*)
equivalent expressions (*87*)
simplifying an algebraic expression (*88*)

2. If R is the set of real numbers, Q is the set of rational numbers, I is the set of integers, P is the set of positive integers, Ne is the set of negative integers, N is the set of natural numbers, and W is the set of whole numbers, replace ● with $\subseteq$ or $\nsubseteq$ to make each of the following sentences true.

a. I ● Q **b.** P ● Q **c.** P ● N
d. R ● Q **e.** P ● I **f.** Ne ● I
g. W ● I **h.** P ● W **i.** W ● R

3. Using the same sets as in Exercise 2, replace ● with $\in$ or $\notin$ to make each of the following sentences true.

a. $\sqrt{2}$ ● Q **b.** $-\frac{2}{3}$ ● R **c.** $-\frac{5}{8}$ ● I
d. 6 ● N **e.** 0 ● N **f.** $\frac{7}{6}$ ● W
g. $\sqrt{36}$ ● I **h.** $\sqrt{10}$ ● R **i.** $|4|$ ● P

4. If x is replaced by -3, what is the value of each of the following?

a. $|x|$ **b.** $|-x|$ **c.** $-x$ **d.** x

5. Draw a number line and a single vector to show the result of each of the following, starting from zero.

a. A gain of 3 and a loss of 5.
b. A gain of 2 and a gain of 6.
c. A loss of 6 and a gain of 9.
d. A loss of 3 and a loss of 4.

6. Give an equation that illustrates each of the following postulates for addition of real numbers.

a. Inverse postulate. **b.** Commutative postulate.
c. Associative postulate. **d.** Identity postulate.

7. Name each sum with a single numeral.

a. $8 + 6$ **b.** $9 + (-2)$ **c.** $(-12) + (-10)$
d. $-8.96 + 5.24$ **e.** $(-\frac{2}{3}) + (-\frac{5}{6})$ **f.** $-76 + 76$

8. Add -3 to the sum of each of the following.

a. $-7 + 5$ **b.** $(-13) + (-2)$ **c.** $8 + 19$

9. What number would replace n to make each sentence true?

a. $7 + n = 13$ **b.** $7 + n = 3$
c. $-7 + n = 3$ **d.** $-7 + n = -3$

10. Name each difference with a single numeral.

a. $18 - 6$ **b.** $-9 + (+9)$ **c.** $-8 + (|-13|)$
d. $5\frac{3}{4} + (-7\frac{1}{2})$ **e.** $-38 + 47$ **f.** $123 + (+154)$

11. Give an equation to illustrate each of the following postulates for multiplication of real numbers.

a. Inverse postulate. **b.** Commutative postulate.
c. Associative postulate. **d.** Identity postulate.

12. Find the products of the following numbers.

a. -7 and -4 **b.** -8 and 5 **c.** 7 and 6
d. $-\frac{5}{8}$ and 0 **e.** $-\frac{7}{8}$ and $-\frac{8}{7}$ **f.** $\frac{3}{5}$ and 1

13. Multiply each product by -10.

a. $(-13)(-6)$ **b.** $(-11)(5)$ **c.** $(18)(3)$

14. What number should replace n to make each sentence true?

a. $-6n = -24$ **b.** $-9n = 45$

c. $4n = -28$ **d.** $6n = 72$

15. Divide as indicated.

a. $\frac{-10}{5}$ **b.** $\frac{-9}{-3}$ **c.** $\frac{18}{-6}$ **d.** $\frac{26}{13}$ **e.** $\frac{-3.9}{-1.3}$

16. Let $x = -3$, $y = -15$, and $z = 5$. What is the value of each of the following?

a. $x + y + z$ **b.** $\frac{x}{y}$ **c.** yz **d.** $x - z$ **e.** $\frac{x+y}{z}$

17. Simplify each of the following.

a. $3x + 7x$ **b.** $5x - 9x$ **c.** $4x + 2y + 3x + 7y$

d. $-8(3x + y)$ **e.** $-3xy + 2 - 7xy$ **f.** $-5(x + y) + 6(2x - y)$

What postulates for the real numbers should be given as reasons for each of the following?

18. $(-15) + (19) = (-15) + (15 + 4)$

a. $= (-15 + 15) + 4$?

b. $= 0 + 4$?

c. $= 4$?

19. $n + (-9) = 2$

a. $[n + (-9)] + 9 = 2 + 9$?

b. $n + [(-9) + 9] = 2 + 9$?

c. $n + 0 = 2 + 9$, or $n = 11$?

20. You know that $5 \cdot 6 = 30$ is true. If the sum of $5 \cdot 6$ and $5 \cdot (-6)$ is 0, then $5(-6)$ is equal to ?. Why?

CHAPTER TEST

Name each of the following with a single numeral.

1. $-8 + 5$ **2.** $-9 + 12$

3. $-6\frac{1}{2} + (-3\frac{1}{2})$ **4.** $-7 - (+5)$

5. $-8\frac{1}{2} + (+6\frac{1}{4})$ **6.** $(-5) \cdot (-6)$

7. $(+8) \cdot (3\frac{1}{2})$ **8.** $\frac{-30 - 2}{-7}$

9. $(-1)^2(-2)(-3)$ **10.** $(-5)(-8)(-2)[6 + (-6)]$

Write *True* or *False* to describe each of the following sentences.

11. $8(-3 + 5) = -24 + 40$ **12.** $(15 - 6) - 3 = 15 - (6 - 3)$

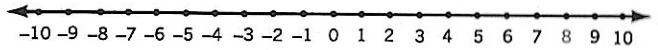
13. $+9 + (-9) = 0 \cdot |-6|$ **14.** $-|-2| < \frac{1}{2}$

15. $(-2 \cdot -7) - (-5) = 9$ **16.** $-16(\frac{3}{4}) = \frac{1}{2}(18) + 3$

17. $\frac{2}{5} \cdot 2\frac{1}{2} = 5\frac{1}{3} \cdot \frac{3}{16}$ **18.** $(+3 + -1) - (-|+4|) = 4$

19. $\frac{2 + (-2)}{15} = 0$ **20.** $\frac{|+6|}{-11} < 0$

Where do you stop on the number line if you start at zero and make successive moves as indicated in each of the following?



21. $+6, -4, -7, +5$

22. $-8, +7, -2, -6, +5, -9, 0, +6$

23. $-8, -2, +6, -4$

24. $-7\frac{1}{2}, +6, +3\frac{1}{2}, -\frac{1}{2}, +2$

Simplify each of the following expressions.

25. $7k + 4k + 3k$

26. $7mn + 4nm + 3m^2n$

27. $3km + 2m + m$

28. $10m + 2m - 3m + 6n + 4m - 4n$

29. $\frac{2m}{3km} - \frac{1}{k} + \frac{n^2}{2k}$

30. $3km + 4k^2m - k^2m + 5km + 4m - 2k$

If $k = 1$, $m = -2$, and $n = \frac{1}{2}$, evaluate the following expressions.

31. $8m^2 + 2k - n$

32. $5k - 8m + 3m + n^3$

33. Mr. Kearney bought 5 shares of stock at $\$25\frac{1}{2}$ each. The stock went up $\$1\frac{1}{4}$, up $\$\frac{3}{8}$, down $\$\frac{1}{2}$, and up \$1 before he sold it. What total profit did he make?

34. Suppose you walk to the top of a hill that is 150 feet above sea level. Walking on a winding path down the other side of the hill, you go down 40 feet, up 10 feet, down 15, and up 2. How much farther is it down to sea level?

35. In a series of plays, a football team made two advances of 7 yards, a loss of 5, a gain of 12, and 2 losses of 3. Express the total change in its position as one number.

Which postulate or postulates justify the following true sentences? The replacement set is the set of real numbers.

36. $\frac{1}{x}(x + y) = 1 + \frac{y}{x}$

37. $ab + ac = a(c + b)$

38. $\left(s \cdot \frac{1}{s}\right)t = t$

39. $a[(b + c)]\frac{1}{d} = [a(b + c)]\frac{1}{d}$

40. $b(a - a) = b \cdot 0$

41. $-x(1 - y) = -x + xy$

ENRICHMENT

Equivalent Infinite Sets

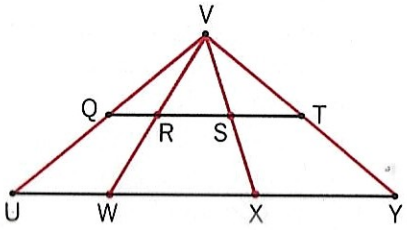
A finite set can be counted, with the counting coming to an end. The set of names of days of the week is an example of a finite set.

Some infinite sets can also be counted, but the counting does not come to an end. If an infinite set can be put in one-to-one correspondence with the natural numbers, it is countably infinite, or **countable.** The following sets are countable and they are equivalent.

Natural numbers	$\{1, 2, 3, 4, 5, \cdots, n, \cdots\}$
Even whole numbers	$\{0, 2, 4, 6, 8, 10, \cdots, 2n, \cdots\}$
Multiples of 10	$\{0, 10, 20, 30, 40, 50, \cdots, 10n, \cdots\}$

Are all infinite sets countable? No, the real numbers are not countable, and the points on a line are not countable. Can they be put in one-to-one correspondence? Yes. As you know from the beginning of this chapter, the set of real numbers and the points on a line are equivalent; they can be put in one-to-one correspondence.

Now look at some other examples of infinite sets. Do you think there are as many points on a line segment one inch long as there are on a line segment two inches long? It would seem that the number of points is different, but look at the following construction.



Segment QT is one inch long, and segment UY is two inches long. A one-to-one correspondence can be established between the points of the two line segments. Some of the correspondences are as follows.

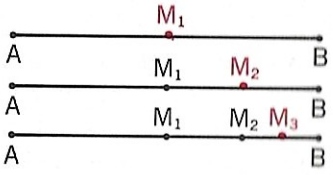
$$Q \leftrightarrow U \qquad S \leftrightarrow X$$
$$R \leftrightarrow W \qquad T \leftrightarrow Y$$

Describe the procedure by which the four correspondences were made. Can all the points on $\overline{QT}$ be matched with all the points on $\overline{UY}$ by the same procedure? Then do $\overline{QT}$ and $\overline{UY}$ contain the same number of points?

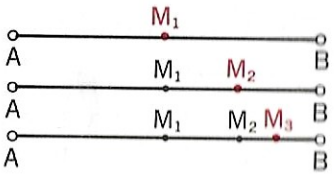
Suppose you consider a line segment AB and the same line segment without the endpoints, designated by $(\overline{AB})$. Can you establish a one-to-one correspondence between $\overline{AB}$ and $(\overline{AB})$?

Just in thinking about setting up the correspondence, it would seem that you would make a correspondence between the points of both segments and have A and B left over. But a similar case would be setting up a one-to-one correspondence between the whole numbers and the natural numbers; that is, the whole numbers are the natural numbers plus zero, and the sets are still equivalent; zero is not left over. So how can you show that $\overline{AB}$ and $(\overline{AB})$ are equivalent? One way to do it is to construct a countable subset of each of the uncountable sets and to form a correspondence between those. Follow the procedure below. (*Note:* $\overline{M_1B}$) may be interpreted as a line segment with M_1 as one endpoint, but no endpoint on the other end.)

① Let M_1 be the midpoint of $\overline{AB}$.
Let M_2 be the midpoint of $\overline{M_1B}$.
Let M_3 be the midpoint of $\overline{M_2B}$.
.
.
.



② Let M_1 be the midpoint of $(\overline{AB})$.
Let M_2 be the midpoint of $\overline{M_1B}$).
Let M_3 be the midpoint of $\overline{M_2B}$).
.
.
.



③ Let Q be the set of points on $(\overline{AB})$ other than the set $\{M_1, M_2, M_3, \cdots\}$. Then $(\overline{AB})$ is $Q \cup \{M_1, M_2, M_3, \cdots\}$ and $\overline{AB}$ is $Q \cup \{A, B, M_1, M_2, M_3, \cdots\}$.

(Continued on next page.)

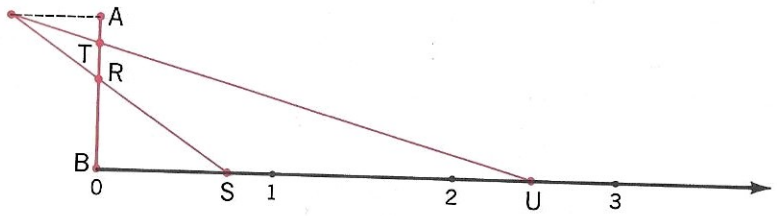
④ Therefore, Q can be put in one-to-one correspondence with itself and $\{M_1, M_2, M_3, \cdots\}$ can be put in one-to-one correspondence with $\{A, B, M_1, M_2, M_3, \cdots\}$.

$$\begin{array}{ccccccccc} Q & \cup & \{M_1, & M_2, & M_3, & M_4, & \cdots\} \\ \updownarrow & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ Q & \cup & \{A, & B, & M_1, & M_2, & \cdots\} \end{array}$$

Thus, the two sets are equivalent and $\overline{AB}$ and $(\overline{AB})$ have the same number of points.

Exercises

1. Are there as many points on a line segment one unit long as there are on the entire number line extending from 0 to the right indefinitely? You need to show a one-to-one correspondence between the segment and the number line. Can you explain this correspondence by studying the figure below?



2. Think back to the problem about the one-to-one correspondence between the segment with endpoints (closed segment) and the segment without endpoints (open segment). Does it matter in Exercise 1 that the endpoint of the segment of 1 unit does not have a corresponding point on the number line by that construction?

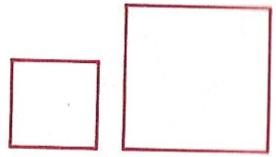
3. What relationship exists between the points of any two lines or line segments?

4. For the figure shown at the right, let P be the set of points on rectangle $ABCD$. Let Q be the set of points on triangle ABC. Then classify each of the following as true or false. Explain.

a. $P \subset Q$ **b.** $P = Q$

c. P is equivalent to Q. **d.** $Q \subset P$

5. Perhaps you can see how these ideas can be extended beyond straight lines. Can you think of a way to show that there are as many points on the large square shown as on the smaller one? You may want to use three-dimensional space. Think of the squares as pieces of cardboard that you are holding parallel to one another.



THE GROWTH OF MATHEMATICS

Mathematics Between the Tigris and the Euphrates

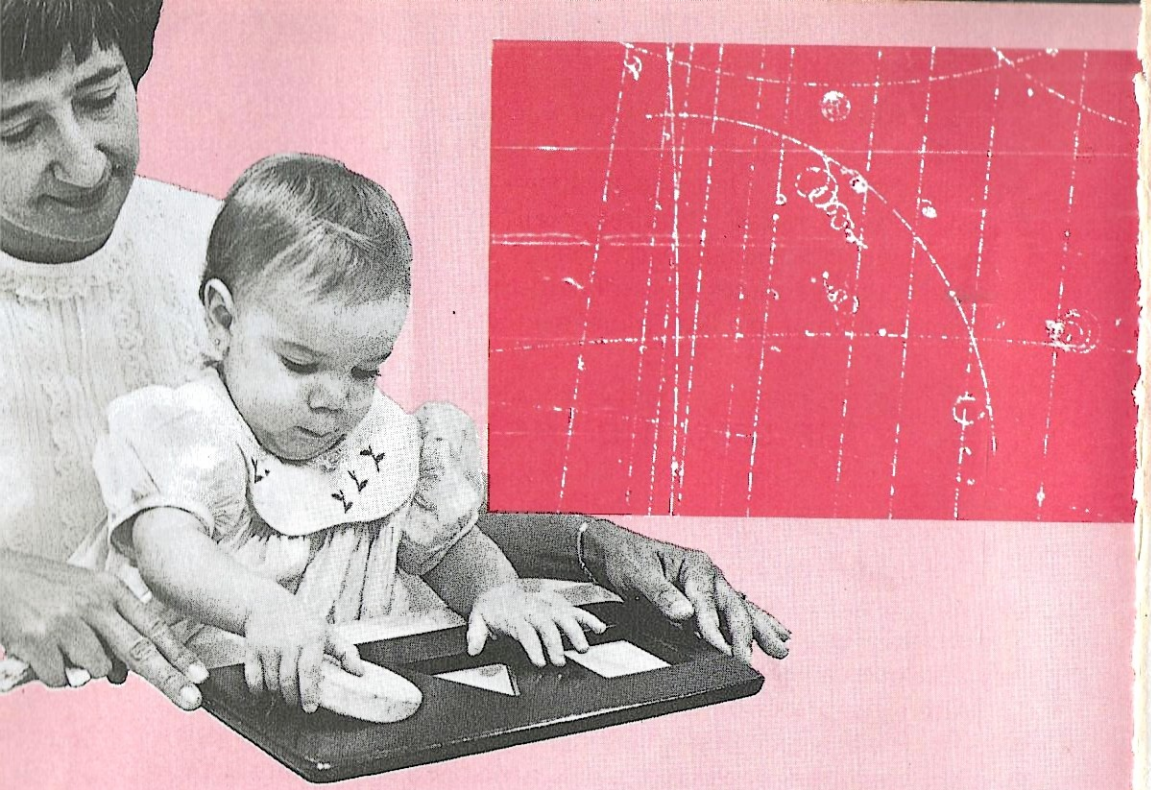
The Sumerians, the first people and culture to develop writing, flourished around 3500 B.C. in a region now known as Iraq, situated between the Tigris and the Euphrates rivers. In addition to writing, they developed a versatile system of numeration with 60 as its base.

Although various other peoples occupied this region through the years, there was a good deal of continuity in the growth of mathematics. Over the years, the peoples of the land added new refinements to their numeration system and progressed in the development of algebra. It was the Babylonians, the great astronomers, who attained the highest level of civilization in this one region.

Mathematics was still practical, with little consideration given to theory. Since much of the computation was done by the tedious process of trial and error, the Babylonians kept records of their results and became great compilers of tables. Of the thousands of Babylonian clay tablets available today, nearly two hundred are mathematical tables of all kinds: multiplication tables, tables of squares and square roots, cubes and cube roots, reciprocals, exponents, and so on.

Another one hundred tablets are problem texts. On them are explicit procedures for solving problems, followed by many more problems arranged in order of difficulty, indicating that they were probably used for teaching. The problems were of many different types, illustrating that they could solve many algebraic equations, including such equations as $x^2 + x = 6$, which you will learn to solve later in the book. But with all their advancement of algebra, they still did not state solutions in general terms.

The Babylonians were not very skilled in geometry. They used 3 as the value for π. Their formulas for the area of a quadrilateral and for the volume of the frustum of a cone and a pyramid were incorrect. Their main contributions were in the number system and algebra.



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CHAPTER 3

SENTENCES AND PROBLEM SOLVING

Many problems can be solved with the aid of mathematics, although not all of these problems are mathematical ones. A mathematical sentence, that is, an equation or an inequality, is often used as a mathematical model for solving problems. The solution of the mathematical sentence is the solution to the problem when the sentence is a faithful model of the problem.

- In the upper left photograph a child is taking a test. The blocks must be placed in the holes in the board. The hole is similar to a mathematical sentence. When the correct one from the set of blocks is chosen, the block fits and the problem is solved. In solving a sentence, when the correct number from the replacement set is chosen, the sentence is true and the problem is solved.
- The lever in the lower right photograph can be described by the formula $lw = LW$. It does not describe the physical properties of the lever, but for solving problems about levers, it is a satisfactory model.
- When equations are complicated, it is helpful to find simpler, equivalent sentences. Experimenting with a balance scale can suggest the principles for obtaining sentences that are equivalent to a given sentence. If the scale is already in balance, for example, what happens when you add equal weights to the two sides?
- The upper right photograph shows the tracks made by subatomic particles in the field of a powerful magnet. Physicists identify the type of particle from two conditions: the length of the path and how the magnet affects the path. Each condition is needed for identification. Many problems involve more than one condition.

3–1 Equivalent Sentences

Recall that $3a + 2a$ and $5a$ are equivalent expressions if they represent the same number for all numbers that may replace a. **Equivalent sentences** has a slightly different meaning from *equivalent expressions*. Can you guess what *equivalent sentences* means?

In the table below, the sentence in column B is equivalent to the corresponding sentence in column A. Check each sentence in B to see if it has the same solution set as the sentence in that row in column A. Recall that the *solution set* is the set of numbers from the replacement set that make the sentence true. In this case the replacement set is the set of integers.

	A	Solution set	B	Solution set
a.	$3x = -9$	$\{-3\}$	$15x = -45$	?
b.	$15 + 2x = 5x$	$\{5\}$	$15 = 3x$	?
c.	$5\frac{1}{2} = n - 9$	$\varnothing$	$11 = 2n - 18$	?
d.	$x + 2 > 5$	$\{4, 5, 6, 7, \cdots\}$	$2x + 4 > 10$	?
e.	$8 - y = 3$	$\{5\}$	$-y = 3 - 8$	?
f.	$2x + 3x = 10$	$\{2\}$	$5x = 10$	?

Now you should be able to understand the definition of equivalent sentences.

Definition Two sentences are equivalent if and only if they have the same solution set.

One way to find an equivalent sentence is by replacing an expression by an equivalent expression. Suppose, however, you have the equation $x + 7 = 15$. Then, this method will not help. You know, however, that $8 + 7 = 15$, so the solution set of $x + 7 = 15$ is $\{8\}$ when the replacement set is the set of integers. Therefore, $x + 7 = 15$ and $x = 8$ are equivalent sentences.

Using the simple example above, you can discover a procedure for finding the solution set of certain equations.

	$x + 7 = 15$
Add -7 to $x + 7$ and -7 to 15.	$x + 7 + (-7) = 15 + (-7)$
Find equivalent expressions.	$x + 0 = 8$
	$x = 8$

By adding -7 to $x + 7$ and -7 to 15, you get the equation $x = 8$, whose solution set, obviously, is $\{8\}$. Do you think that you will always get an equivalent equation if you add the same number to both sides of any equation? Well, you will! This property is so important and so useful that it is stated as a postulate.

Postulate 3-1 (Addition Postulate for Equations) If the same number is added to both sides of an equation, the result is an equivalent equation.

If $a = b$ is true, then $a + c = b + c$ is true.

Now try these

— For each exercise, answer these questions: What is added to the left side of equation **a** to get the left side of equation **b**? What is added to the right side of **a** to get the right side of **b**? Is equation **b** equivalent to equation **a**?

Be able to give an explanation for each answer.

1. **a.** $x - 4 = 15$
b. $x - 4 + 4 = 15 + 4$

2. **a.** $x - 8 = 6$
b. $x - 8 + 3 = 6 + 2$

3. **a.** $4x + 12 = 6$
b. $4x + 12 + (-12) = 6$

4. **a.** $\frac{3}{4}x - 17 = 24$
b. $\frac{3}{4}x = 17 + 24$

5. **a.** $18 + 3x = 6 + 7x$
b. $18 + 3x + (-3x) = 6 + 7x + (-3x)$

— What must be added to the expression in **a** to get the expression in **b**?

6. **a.** $x + 7$
b. x

7. **a.** $3x - 9$
b. $3x$

8. **a.** $8 - 7x$
b. $-7x$

9. **a.** $3t + 0.2$
b. $3t$

Answers: **1.** 4, 4, yes. **2.** 3, 2, no. **3.** -12, 0, no. **4.** 17, 17, yes. **5.** $-3x$, $-3x$, yes. **6.** -7 **7.** 9 **8.** -8 **9.** -0.2

Checkpoint

1. What are equivalent expressions? Give an example.
2. What are equivalent equations? Give an example.
3. State the addition postulate for equations in your own words.

Exercises

A — Is sentence **a** equivalent to sentence **b**?

1. **a.** $x + 4 = 9$
   b. $x + 4 - (-4) = 9 + (-4)$
2. **a.** $y - 9 = 3$
   b. $y - 9 + 9 = 3$
3. **a.** $2n + 9 = 11$
   b. $2n = 11 + (-9)$
4. **a.** $t = 6$
   b. $t + 6 = 12$
5. **a.** $3x - 8 = 2x + 5$
   b. $3x - 8 + 7 = 2x + 5 + (-5)$
6. **a.** $9 = 11 - 4m$
   b. $20 = -4m$
7. **a.** $17 - 6x = 22$
   b. $17 = 22 + 6x$
8. **a.** $17 - 6x = 22$
   b. $-6x = 39$
9. **a.** $2(3x + 8) - 2x = 23$
   b. $4x + 16 = 23$
10. **a.** $5n - 3(7 - 2n) = n + 2$
    b. $12n = 23$

— For the following exercises, use the addition postulate for equations to write an equivalent equation with only the variable on one side. Use braces to show the solution set after you have checked the solution in the original equation. The replacement set is the set of real numbers.

Example. Solve $x + 9 = 12\frac{1}{2}$ where the replacement set for x is the set of real numbers.

Add -9 to both sides. $x + 9 + (-9) = 12\frac{1}{2} + (-9)$

Find equivalent expressions. $x = 3\frac{1}{2}$

To check, replace x in the original equation with $3\frac{1}{2}$.

Check $3\frac{1}{2} + 9 = 12\frac{1}{2}$

$12\frac{1}{2} = 12\frac{1}{2}$

Solution set: $\{3\frac{1}{2}\}$

11. $x + 7 = 13$
12. $8 + n = 5$
13. $m + 3\frac{1}{2} = 7$
14. $t + 11.2 = 3.7$
15. $14 = y - 6$
16. $t - 4 = 9$
17. $8\frac{1}{2} = x + 13\frac{1}{8}$
18. $x - 6\frac{1}{4} = 12\frac{1}{2}$
19. $3 - y = 17$
20. $5.4 - n = -0.6$

B — Solve each equation below. The replacement set is the set of integers.

Example. $3x - 9 = 2x + 12$

Add $-2x$. $3x + (-2x) - 9 = 2x + (-2x) + 12$

$x - 9 = 12$

Add 9. $x - 9 + 9 = 12 + 9$

$x = 21$

Check $3(21) - 9 = 2(21) + 12$

$63 - 9 = 42 + 12$

$54 = 54$

Solution set: $\{21\}$

21. $3x + 5 = 2x - 6$
22. $4 - 7n = 1 - 6n$
23. $\frac{1}{2}y - \frac{3}{4} = \frac{1}{4} - \frac{1}{2}y$
24. $4m + 9 = 9 + 5m$
25. $3n + 4 = 2n + 8$
26. $5m + 12 = 7 + 6m$
27. $8m - 9 = 7m + 5$
28. $m = 2m - 9$
29. $4m - 15 = 5m - 9$
30. $5x - 5\frac{1}{2} = 4x + 5\frac{1}{2}$

C 31. Prove that if $a + c = b + c$ is true, then $a = b$ is true. How is this different from the addition postulate for equations?

3–2 The Multiplication Postulate for Equations

You may use a similar pattern of thinking for multiplication and addition when you want to find an equivalent sentence to help in finding a solution to an equation. Start with an equation such as $\frac{3}{4}x = 15$ and multiply both sides by $\frac{4}{3}$. You get the new equation $\frac{4}{3}(\frac{3}{4}x) = \frac{4}{3}(15)$, or $x = 20$. If 20 replaces x, will both equations be true? Is the solution set of $\frac{3}{4}x = 15$ and of $x = 20$ the same? Are the equations equivalent?

The idea illustrated by this example can be stated as a postulate.

Postulate 3–2 (Multiplication Postulate for Equations) If both sides of an equation are multiplied by any nonzero number, the result is an equivalent equation.

If $a = b$ is true and $c \neq 0$, then $ac = bc$ is true.

The multiplication postulate for equations can be used to solve an equation such as $-8w = 80$. Since the product of any number and its multiplicative inverse is 1, it is necessary to multiply $-8w$ by $-\frac{1}{8}$ to give a product of $1 \cdot w$. If you multiply $-8w$ by $-\frac{1}{8}$, by what number must you multiply 80? The sentence obtained when both sides of the equation $-8w = 80$ are multiplied by $-\frac{1}{8}$ is $w = -10$. Show that $w = -10$ is equivalent to $-8w = 80$.

EXAMPLE. Solve $-4a = 20$. The replacement set is the set of integers.

Multiply both sides by $-\frac{1}{4}$. $\quad -\frac{1}{4}(-4a) = -\frac{1}{4}(20)$

$a = -5$

Check $\quad -4(-5) = 20$

Solution set: $\{-5\}$

Now try these

— In each exercise, answer these questions: By what is the left side of equation **a** multiplied to get the left side of **b**? By what is the right side of **a** multiplied to get the right side of **b**? Is equation **b** equivalent to equation **a**?

1. a. $\frac{1}{3}x = 9$
b. $3(\frac{1}{3}x) = 3(9)$

2. a. $\frac{5}{2}y = 10$
b. $\frac{2}{5}(\frac{5}{2}y) = \frac{2}{5}(10)$

3. a. $\frac{x}{2} = \frac{5}{2}$
b. $x = 3(\frac{5}{2})$

4. a. $0.05x = 3$
b. $20(0.05x) = 20(3)$

— By what must you multiply expression **a** to get expression **b**?

5. a. $3x$ **b.** $1(x)$

6. a. $-2x$ **b.** $1(x)$

7. a. $0.02n$ **b.** n

8. a. $\frac{2}{3}n$ **b.** $1(n)$

Answers: **1.** 3, 3, yes. **2.** $\frac{2}{5}$, $\frac{2}{5}$, yes. **3.** 2, 3, no. **4.** 20, 20, yes. **5.** $\frac{1}{3}$ **6.** $-\frac{1}{2}$ **7.** 50 **8.** $\frac{3}{2}$

Exercises

A — Use the multiplication postulate for equations to find the solution set of each. The replacement set is the set of real numbers. Check.

1. $2x = 8$ **2.** $\frac{1}{4}x = 7$ **3.** $\frac{2}{3}n = 12$
4. $\frac{1}{2}y = -4$ **5.** $3x = 19$ **6.** $\frac{2}{3}m = -8$
7. $-2x = 12$ **8.** $-\frac{3}{4}y = 27$ **9.** $-5y = -8$
10. $10 = -\frac{3}{4}n$ **11.** $\frac{2}{3} = 5t$ **12.** $-9 = \frac{5}{4}v$

— Use the addition or the multiplication postulate to find the solution set of each. The replacement set is the set of real numbers. Check.

13. $n + 3\frac{1}{2} = 9\frac{3}{4}$ **14.** $n - \frac{1}{2} = 4$ **15.** $3\frac{1}{4} = x + 2\frac{1}{2}$
16. $-9a = 4$ **17.** $\frac{x}{1.4} = 4.5$ **18.** $-6x = 0.48$
19. $n + 7\frac{1}{2} = 9\frac{2}{3}$ **20.** $\frac{x}{2} = 2\frac{1}{2}$ **21.** $\frac{n}{2.4} = 0.06$
22. $-13\frac{1}{2}a = 27$ **23.** $2.3 = 4.9 + t$ **24.** $4 = \frac{8}{3}v$

— Recall that the formula $\vec{d} = \vec{v}t$ gives the distance $\vec{d}$ traveled in a given direction (positive or negative) if the velocity $\vec{v}$ and the time t are known. Write an equation for each exercise and solve.

25. $\vec{v} = +10$ miles per hour; $t = 5$ hours; find $\vec{d}$.
26. $\vec{v} = -10$ miles per hour; $t = 7$ hours; find $\vec{d}$.
27. $\vec{d} = 120$ meters; $\vec{v} = 30$ meters per second; find t.
28. $\vec{d} = 352$ feet; $t = 4$ seconds; find $\vec{v}$.
29. $\vec{d} = -\frac{2}{3}$ kilometer; $t = 240$ seconds; find $\vec{v}$.
30. $\vec{d} = -2.5$ kilometers; $\vec{v} = -40$ kilometers per hour; find t.
31. $\vec{d} = -0.06$ meter; $\vec{v} = -1.2$ meters per second; find t.
32. $\vec{d} = -55.4$ centimeters; $t = 5.2$ seconds; find $\vec{v}$.

B — Write *True* or *False* to describe each of the following sentences.

33. $\forall x \;\; x + 2 = 3$ **34.** $\forall x \;\; x = x + 1$
35. $\forall x \;\; 3x + 2x = 5x$ **36.** $\forall x \;\; \frac{x}{2} = \frac{1}{2}x$

— Find the solution set of each equation if the replacement set is the set of whole numbers. Remember that the solution set is empty if no whole number gives a true sentence.

37. $\frac{1}{4}x = 20$ **38.** $\frac{1}{3}t = 2$ **39.** $\frac{1}{2}x = -4$
40. $\frac{2}{3}x = 1$ **41.** $\frac{3}{4}n = 16$ **42.** $2.5y = 6.25$

3–3 Using Two Postulates to Solve Equations

Sometimes it is necessary to find several equivalent equations before you find one that has the variable alone on one side. For example, to solve an equation such as $3x + 8 = 23$, you must use both postulates for equations. The postulates for equations and the other postulates about real numbers that you have studied in earlier chapters

enable you to give reasons for each step in finding a chain of equivalent equations. Look at the way the equation $3x + 8 = 23$ can be solved.

	$3x + 8 = 23$
Add -8 to both sides, using the addition postulate for equations.	$3x + 8 + (-8) = 23 + (-8)$
Use additive inverse postulate and addition of real numbers.	$3x = 15$
Multiply both sides by $\frac{1}{3}$, using multiplication postulate for equations.	$\frac{1}{3}(3x) = \frac{1}{3}(15)$
Use multiplicative inverse postulate and multiplication of real numbers.	$x = 5$

When 5 replaces x in $3x + 8 = 23$, the true equation $3(5) + 8 = 23$ results. Hence, the solution set is $\{5\}$.

The number that makes the equation true is the **solution** of the equation, or the **root** of the equation. You may also say that 5 *satisfies* the equation.

Look again at the chain of five equivalent equations above. The goal is the last equation, the one whose solution set is obvious.

A different order of steps for solving equations is quite permissible as shown below.

	$3x + 8 = 23$
Multiply both sides by $\frac{1}{3}$.	$\frac{1}{3}(3x + 8) = \frac{1}{3}(23)$
Apply distributive postulate.	$x + \frac{8}{3} = \frac{23}{3}$
Add $-\frac{8}{3}$ to both sides.	$x + \frac{8}{3} + (-\frac{8}{3}) = \frac{23}{3} + (-\frac{8}{3})$
Find equivalent expressions.	$x = \frac{15}{3}$, or 5
Check	$3(5) + 8 = 23$

Solution set: $\{5\}$

Which method of finding a chain of equivalent equations do you prefer?

EXAMPLE. Solve $\frac{3}{5}n - 7 = 14$.

		Check
	$\frac{3}{5}n - 7 = 14$	$\frac{3}{5}(35) - 7 = 14$
Add $+7$.	$\frac{3}{5}n = 21$	$21 - 7 = 14$
Multiply by $+\frac{5}{3}$.	$n = 35$	$14 = 14$

Solution set: $\{35\}$

Now try these

Solve for x.

1. $2x + 4 = 8$ **2.** $3x - 1 = 8$

3. $5 - x = 9$ **4.** $\frac{x}{4} - 2 = -1$

Answers: **1.** $\{2\}$ **2.** $\{3\}$ **3.** $\{-4\}$ **4.** $\{4\}$

Checkpoint

1. Does the order in which the addition and multiplication postulates for equations are applied affect the solution of an equation?

2. What steps other than applying the addition and multiplication postulates for equations may occur in solving an equation such as $5x + 9 = 29$?

Exercises

A Find the solution sets of the following equations. The replacement set is the set of rational numbers. When you use an equation postulate in finding the solution set, indicate which one. Check your results.

1. $5n + 3 = 33$	**2.** $5n - 3 = 17$	**3.** $2n + 3 = 7$
4. $5x - 9 = -19$	**5.** $3t + 17 = 5$	**6.** $4a + 17 = 53$
7. $2n + 5 = 17$	**8.** $3n - 4 = 17$	**9.** $3n - 17 = 4$
10. $3n + 17 = 4$	**11.** $2a + 15 = 3$	**12.** $6a + 1 = 37$
13. $8a - 7 = 41$	**14.** $2b + 3 = 21$	**15.** $4v + 27 = 3$
16. $3a + 5 = 32$	**17.** $9a - 1 = 80$	**18.** $5 - 2y = 15$
19. $7 = 6x + 19$	**20.** $8 = 5t + 33$	**21.** $4 = 30 - 6m$
22. $3b - 5 = 32$	**23.** $6x + 37 = 17$	**24.** $\frac{2}{3}t + 5 = 23$
25. $9a - 8 = 73$	**26.** $\frac{y}{2} + 11 = 5$	**27.** $15 - \frac{5}{4}v = 23$
28. $8n + 22 = 70$	**29.** $1.2t + 3.4 = -1.0$	**30.** $-0.6n + 11 = 17$
31. $2.1 = 4.3 - 1.1w$	**32.** $0.8n + 7.0 = 2.2$	**33.** $0 = 0.6n - 3.6$
34. $\frac{7}{10}n + 6 = 41$	**35.** $\frac{5}{6}n + 34 = 9$	**36.** $\frac{8}{9}n + 12 = 84$
37. $\frac{9}{10}x - 17 = 19$	**38.** $\frac{16}{15}x + 78 = 14$	**39.** $\frac{3}{4}x - 3 = 18$
40. $\frac{7}{8}t - 8 = 34$	**41.** $\frac{2}{3}t - 13 = 57$	**42.** $\frac{5}{12}t - 12 = 48$
43. $28 = \frac{17}{32}n - 23$	**44.** $6 = 26 + \frac{2}{5}n$	**45.** $15 = \frac{3}{10}n - 15$
46. $18 - \frac{9}{32}x = 27$	**47.** $53 + \frac{3}{5}x = 26$	**48.** $49 + \frac{3}{5}x = 16$

B — Solve and check the following equations. The replacement set is the set of real numbers.

49. $32{,}845 + 49x = 41{,}284$

50. $-271 + 33t = -485$

51. $\frac{2}{3} = \frac{3}{4}y - \frac{5}{6}$

52. $-\frac{4}{5} = \frac{7}{12} - \frac{w}{3}$

53. $0.6x + 0.06 = 0.006$

54. $0.12 - 0.35x = 0.049$

55. $2\pi + 7x = 5\pi$

56. $9 + 4x = 5\pi$

3–4 Sets and Existence

You have learned several ways to describe a set. Explain why the following are all descriptions for the same set.

a. The set of even whole numbers less than twenty.

b. {the even whole numbers less than twenty}

c. {0, 2, 4, 6, 8, 10, 12, 14, 16, 18}

d. {0, 2, 4, · · · , 16, 18}

There is another method of describing a set that is frequently convenient. A variable may be used together with a rule. For example, the set described above in **a–d** could also be given as follows.

e. The set of all whole numbers that replace n so that n is even and less than 20.

The description in **e** has a simple abbreviation, as shown in **f**.

f. $\{n : n \text{ is an even whole number less than } 20\}$

The colon is read as "such that," so **f** may be read "The set of all numbers n such that n is an even whole number less than 20."

The set description in **f** is an example of **set-builder notation.** Usually equations and other mathematical sentences are used with set-builder notation, rather than word descriptions. For example, an even number is a whole-number multiple of 2, so either of the following descriptions would also name the set described in **a–f.**

g. $\{n : n < 20, n = 2a, a \in \mathrm{W}\}$

h. $\{2n : n < 10, n \in \mathrm{W}\}$

Frequently it is desirable to write statements containing variables that are not open sentences. One way that you have previously used involves $\forall$, "for all." Thus, the sentences $\forall x \;\; x + x = 2x$ and $\forall x \;\; x + 3 = 5$ are not open. Notice that $\forall x \;\; x + x = 2x$ is a true sentence and $\forall x \;\; x + 3 = 5$ is false. There is another method that is also useful. The symbol $\exists$ means "there exists." Thus, the sentence $\exists x \;\; x + 3 = 5$, read "There exists an x such that $x + 3 = 5$ is true," is a true sentence. Is $\exists a \;\; a = a + 1$ true or false?

A sentence such as $\exists x \;\; x + x = 2x$ is a true sentence. Why? Do you think that whenever $\exists$ replaces $\forall$ in a true sentence the result is a true sentence?

Notice that for both $\forall$ and $\exists$ it is often necessary to supply some indication of the replacement set for the variable. For example, $\exists x \in \mathrm{R} \;\; 2x + 4 = 5$ is a true sentence, but $\exists x \in \mathrm{W} \;\; 2x + 4 = 5$ is false. When no replacement set for the variable is indicated, you should assume that the replacement set is the set of real numbers.

The symbols $\forall$ and $\exists$ may both occur in the same sentence. For example, $\forall x \in \mathrm{R} \;\; \exists y \in \mathrm{R} \;\; x + y = 0$ is the statement of one of the postulates for the real numbers. Which one is it? Notice that order is important. The sentence $\exists y \in \mathrm{R} \;\; \forall x \in \mathrm{R} \;\; x + y = 0$ is false because it means "There exists a real number y such that for every real number x, the sum of x and y is zero."

Checkpoint

1. Describe the way you would use set-builder notation to denote the set of odd whole numbers between 0 and 20.

2. Describe two ways in which a sentence that is not open may contain a variable.

Exercises

A — Use the roster method to denote each set below. Check.

1. $\{x : x - 2\frac{1}{2} = 6\}$

2. $\{b : -2\frac{1}{3}b = 7\}$

3. $\{x : x + 5 = 5\}$

4. $\{y : y - 8\frac{4}{5} = 6\frac{2}{3}\}$

5. $\{y : y - 8 = -10\}$

6. $\{x : 0.7x = 0.014\}$

7. $\left\{t : \frac{t}{0.6} = 8\right\}$

8. $\{n : n - 3\frac{1}{4} = 2\frac{1}{3}\}$

9. $\{x : 3x - 5 = -11\}$

10. $\left\{u : 12 = 4 - \frac{x}{3}\right\}$

— Write *True* or *False* to describe each sentence.

11. $\forall x \in \mathrm{R} \;\; x + 3 = 7$

12. $\exists x \in \mathrm{R} \;\; x + 3 = 7$

13. $\forall x \in \mathrm{R} \;\; x = x - 1$

14. $\exists x \in \mathrm{R} \;\; x = x - 1$

15. $\forall x \in R \;\; 5x - 2x = 3x$ **16.** $\exists x \in R \;\; 5x - 2x = 3x$

17. $\exists x \in R \;\; x + 5 = 0$ **18.** $\exists x \in R \;\; 3x = 0$

19. $\exists x \in N \;\; x + 5 = 0$ **20.** $\exists x \in N \;\; 3x = 0$

B **21.** Set-builder notation is a precise way to describe sets, but often the description is harder to read than necessary. For example, the following is a correct description of the set $\{5\}$.

$$\{x : \tfrac{3}{5}x + 9 = 12, x \in R\}$$

What is the easiest name for this set?

$$\{a : a = a + 1, a \in Q\}$$

Use set-builder notation to denote the following sets.

22. $\{3, 6, 9, 12, 15, 18\}$ **23.** $\{\cdots, -4, -3, -2, -1, 0\}$

24. $\{7\}$ **25.** $\emptyset$

C Write *True* or *False* to describe each sentence.

26. $\forall x \in R \;\; \exists y \in R \;\; xy = 1$ **27.** $\exists x \in R \;\; \forall y \in R \;\; xy = 1$

28. $\forall x \in R \;\; \exists y \in R \;\; x + y = x$ **29.** $\exists x \in R \;\; \forall y \in R \;\; x + y = x$

3–5 Solving Equations

By now you should be able to solve fairly complicated equations. All you need to do is simplify each side of the equation and use the postulates for equations.

Your goal is to get the terms containing the variable on one side of the equation. You may choose to do it on the left or on the right.

EXAMPLE 1. Solve $x + (3x - 9) + x + (3x - 9) = 62$.

$$x + (3x - 9) + x + (3x - 9) = 62$$

Simplify the left side. $8x - 18 = 62$

Add 18 to both sides. $8x = 80$

Multiply both sides by $\frac{1}{8}$. $x = 10$

Check $10 + (30 - 9) + 10 + (30 - 9) = 62$

$10 + 21 + 10 + 21 = 62$

$62 = 62$

Solution set: $\{10\}$

EXAMPLE 2. Solve $4x - 5x + 7 = 17 + 9x$.

$$4x - 5x + 7 = 17 + 9x$$

$-x + 7 = 17 + 9x$ Why?

$7 = 17 + 10x$ Why?

Add -17. $-10 = 10x$

Multiply by $\frac{1}{10}$. $-1 = x$

Check $4(-1) - 5(-1) + 7 = 17 + 9(-1)$

$-4 + 5 + 7 = 17 - 9$

$8 = 8$

Solution set: $\{-1\}$

EXAMPLE 3. Solve $4x - 3(x - 2) = 41$.

You will find it easier to use the distributive postulate first. Then simplify the left side.

$$4x - 3(x - 2) = 41$$

$4x - 3x + 6 = 41$ Why?

$x + 6 = 41$ Why?

$x = 35$

Check to see that $\{35\}$ is the solution set.

Now try these

First use the distributive postulate on the left side of the equation. Then solve.

1. $3a + 5a = 8$ **2.** $6a - 2a + 3 = 15$ **3.** $7b + 3b - 2b = 16$

4. $2(x + 3) = 0$ **5.** $5(a + 7) = 25$ **6.** $2x + 0.25x + 0.5x = 0$

7. $8(2 - 6x) = 2$ **8.** $8(m - 9) = 35$ **9.** $-2(-3 - 4x) = -10$

Answers: **1.** $\{1\}$ **2.** $\{3\}$ **3.** $\{2\}$ **4.** $\{-3\}$ **5.** $\{-2\}$ **6.** $\{0\}$ **7.** $\{\frac{7}{24}\}$ **8.** $\{13\frac{8}{8}\}$ **9.** $\{-2\}$

Exercises

A Find the solution set of each of the following equations. The replacement is the set of real numbers.

1. $7x + 3x = -9$ **2.** $2n + 5n = 12$

3. $-3n - 4n = -21$ **4.** $2(x - 3) = -10$

5. $2(x - 5) = -8$ **6.** $2(x - 5) = 3$

7. $3(2x + 1) = 1$ **8.** $4(5x + 7) = 13$

9. $2(3n - 5) = -15$ **10.** $-7b + 4(2b - 3) = 16$

11. $-3x + 6(x - 4) = 9$
12. $15a = 26a + 33$
13. $-5a - 19 = 13a + 17$
14. $63a + 13 = 15a - 11$
15. $9a = 44a - 70$
16. $12a = 19a - 15 - 20$

List the members in these sets. The replacement set is R.

17. $\{x : 8x - 13x = 2x + 56\}$
18. $\{n : -6n + 13 = -5n + 12\}$
19. $\{n : 5n - 20 = -32 - 7n\}$
20. $\{n : -16 - 3n = 5n + 16\}$
21. $\{t : 11 - 2t - 16 = 8t\}$
22. $\{n : 14n = 14n - 5n + 6\}$
23. $\{n : 26n - 9n = 46 + 26n\}$
24. $\{c : 5c - 13 = 6c - 18\}$
25. $\{c : 4c + 12 = 7c + 3\}$
26. $\{x : 5 - 3x = 5x + 21\}$
27. $\{a : 3a - 5 = 4a + 23\}$
28. $\{a : 11a - 5 = 4a - 13\}$
29. $\{r : -5r + 2 = -9r - 16\}$
30. $\{s : -6s = -7 + 2s - 13\}$
31. $\{r : 8r - 7 - 3r = -5r + 3\}$
32. $\{u : 5u - 3u + 9 = 2(6u - 5u) - 1\}$

B Solve the following equations where the replacement set is the set of real numbers.

33. $7x - (x - 4) = 25$
34. $-5y - (2 - y) = 18$
35. $2(x - 3) + 3(x - 2) = 8$
36. $2(2n + 1) - 3(n - 5) = 0$
37. $\frac{1}{2}n + \frac{3}{4}n - n = 2.5$
38. $1.5x - 0.2x + 1.2x = 0.25$
39. $1.3 + 0.4n - 0.03n = 3.52$
40. $c + 5 + 0.05c = 11.3$

3–6 Conditional Equations and Identities

In the last chapter you learned to write equivalent algebraic expressions. You learned, for example, that $3x - 7x + 9x - 4$ is equivalent to $5x - 4$. The two expressions name the same number for all replacements for x, which may be indicated by the following true sentence.

$$\forall x \in R \quad 3x - 7x + 9x - 4 = 5x - 4$$

Such an equation is an **identity.** For any real number that replaces x, the value of the left side is identical to the value of the right side.

Definition An identity is an open equation that is true for every replacement of the variable for which the equation has meaning.

The sentence $\frac{3}{x} = \frac{1}{x}(3)$ has no meaning when x is replaced by 0, but it is true for every other replacement. Thus, $\frac{3}{x} = \frac{1}{x}(3)$ is an identity.

Look at the example below to see what happens when you use the addition and multiplication postulates on an identity.

EXAMPLE. Solve the equation $3x - 7x + 9x - 4 = 5x - 4$.

	$3x - 7x + 9x - 4 = 5x - 4$
Use equivalent expressions.	$5x - 4 = 5x - 4$
Add $-5x + 4$ to each side.	$5x - 4 + (-5x + 4) = 5x - 4 + (-5x + 4)$
Use equivalent expressions.	$0 \cdot x = 0 \cdot x$

Is the last equation equivalent to the first? What replacements for x make $0 \cdot x = 0 \cdot x$ true? Then, do all real numbers satisfy $3x - 7x + 9x - 4 = 5x - 4$?

When you solve an equation and find an equivalent equation that is true for all values of x, the original equation is an identity. In the example, you could go one step further to obtain $0 = 0$, which is obviously a true statement. If you obtain a true statement such as $0 = 0$ or $-4 = -4$ in solving an equation, the equation is an identity. You can think of $-4 = -4$ as $0 \cdot x - 4 = -4$ also.

Why is the solution set of most identities the set of real numbers? When is the solution set of an identity not the set of real numbers?

The main work of this chapter so far has been with sentences such as $2x - 3 = 11$, which are true only for certain real numbers. The equation $2x - 3 = 11$ is true under the condition that x is replaced by 7. Thus, $2x - 3 = 11$ is a **conditional equation.**

Definition A conditional equation is an open equation that is not an identity.

Why must the solution set of a conditional equation whose replacement set is the set of real numbers be a proper subset of the set of real numbers?

Checkpoint

1. What is an identity?

2. What is a conditional equation? What is the characteristic of the solution set for a conditional equation?

Exercises

A — Which are identities and which are conditional equations?

1. $2x + 3 = 3x + 8$
2. $10x - 4 = 7x - 7$
3. $3(x - 1) = x - 3 + 2x$
4. $5x = 3x$
5. $x + 2 = 2 + x$
6. $3x - 5 = x - (9 + 2x)$
7. $4x + 2 + 3x = 2x + 3 + 5x$
8. $8y + 6 = 4(3y + 1\frac{1}{2})$
9. $3t + (7 - 4t) = (3t + 7) - 4t$
10. $\frac{5+3}{x} = \frac{1}{x}(6 + 2)$

B — Write *True* or *False* for each sentence.

11. $\forall x \in R \;\; x + 2 = x$
12. $\forall x \in R \;\; 7x - 2x = 5x$
13. $\forall x \in R \;\; 2x > x$
14. $\forall x \in R \;\; 4x - 15 + 2x = 8x - 10$
15. $\exists x \in R \;\; 2x < x$
16. $\exists x \in R \;\; x + 7 > x$
17. $\exists x \in R \;\; x + 7 < x$
18. $\forall x \in R \;\; |x| + |x| > x$

— Write *All* if the sentence is true for every real number, *Some but not all* if it is true for some real numbers but not for all of them, and *None* if it is true for no real number.

19. $4(x - 5) = -20 + 4x$
20. $8(y + 2) - 2(y - 4) = 6$
21. $x + x < x$
22. $-x < x$
23. $|x| < x$
24. $|x + 2| < |x| + 2$

— The following exercise shows what happens when you use the postulates with an equation whose solution set is empty.

25. Supply reasons for each step in rewriting the equation.

$$3(x + 1) = 3(x - 3\tfrac{2}{3})$$
$$3x + 3 = 3x - 11$$
$$0 \cdot x = -14, \text{ or } 0 = -14$$

26. Is each equation in Exercise 25 equivalent to the first? Will any number make $0 \cdot x = -14$ true?

C 27. Think of a number and double it. Add 4 to the result. Divide by 2 and then subtract 2. You have the number you started with.

Number you started with.	x
Double it.	$2x$
Add 4.	$2x + 4$
Divide by 2.	$\frac{2x+4}{2}$
Subtract 2.	$\frac{2x+4}{2} - 2$

Show that $\frac{2x+4}{2} - 2 = x$ is an identity.

28. Think of a number. Multiply it by 4 and then add 8. Divide the result by 2 and then subtract twice the original number. Your result is 4. Explain why.

29. What is the graph of an identity?

3–7 Solving and Graphing Inequalities

Is $\{4\}$ the solution set of the equation $2x + 3 = 11$ if the replacement set for x is the set of integers? Then what do you think is the solution set of the inequality $2x + 3 > 11$? of the inequality $2x + 3 < 11$?

The following table shows the number that results from $2x + 3$ when x is replaced by integers. Cover the values for $2x + 3$ in the table, and find them yourself. Then check your results.

When x is	-1	0	1	2	3	4	5	6	7	8
Then $2x + 3$ is	1	3	5	7	9	11	13	15	17	19

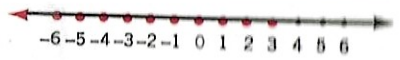
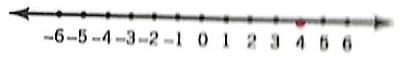
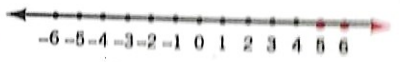
Note the following.

① The number 4 makes $2x + 3$ equal to 11.

② Numbers less than 4 make $2x + 3$ less than 11.

③ Numbers greater than 4 make $2x + 3$ greater than 11.

The solution sets and graphs of $2x + 3 < 11$, $2x + 3 = 11$, and $2x + 3 > 11$ are shown below for x, when x is an integer.

Sentence	Solution set	Graph
$2x + 3 < 11$	$\{\cdots, -1, 0, 1, 2, 3\}$	
$2x + 3 = 11$	$\{4\}$	
$2x + 3 > 11$	$\{5, 6, 7, 8, \cdots\}$	

The graph of each inequality is the set of points extending without bound on one side of the point for 4. Which sentence has a graph that extends to the left? Which sentence has a graph that extends to the right?

The **comparison postulate** insures that the kind of reasoning indicated may be used with many different kinds of inequalities.

Postulate 3–3 (Comparison Postulate) For any two real numbers a and b, one and only one of the following is true.

$$a < b \qquad a = b \qquad a > b$$

EXAMPLE 1. Solve and graph each sentence if x is an integer.

a. $-2x < -2$ **b.** $-2x = -2$ **c.** $-2x > -2$

By the comparison postulate, any number you choose for x will make one and only one of the sentences true.

First, find the solution set for $-2x = -2$.

$$-2x = -2$$
$$x = 1$$

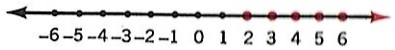
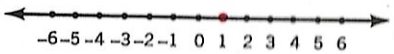
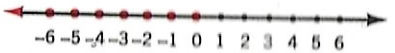
Solution set: $\{1\}$

Then see if numbers less than 1 satisfy $-2x < -2$ or $-2x > -2$. Cover the values for $-2x$ in the table and find them first yourself. Then check your results against the table. (The numbers less than 1 and the product numbers that are less than -2 are shown in color.)

When x is	-3	-2	-1	0	1	2	3	4	5
Then $-2x$ is	6	4	2	0	-2	-4	-6	-8	-10

① Numbers *greater than* 1 make $-2x < -2$ true. The solution set for $-2x < -2$ is $\{2, 3, 4, 5, \cdots\}$.

② Numbers *less than* 1 make $-2x > -2$ true. The solution set of $-2x > -2$ is $\{\cdots, -3, -2, -1, 0\}$.

Sentence	Solution set	Graph
$-2x < -2$	$\{2, 3, 4, 5, \cdots\}$	
$-2x = -2$	$\{1\}$	
$-2x > -2$	$\{\cdots, -3, -2, -1, 0\}$	

If the replacement set were the set of real numbers, how would the graphs of the solution sets differ from the graphs above?

EXAMPLE 2. Solve $3 - 2x > 9$ and graph the solution set. The replacement set for x is the set of real numbers.

① Solve the equation $3 - 2x = 9$.

$$\begin{aligned} 3 - 2x &= 9 \\ -2x &= 6 \quad \text{Why?} \\ x &= -3 \quad \text{Why?} \end{aligned}$$

Check $3 - 2(-3) = 9$
$3 + 6 = 9$
$9 = 9$

Solution set: $\{-3\}$

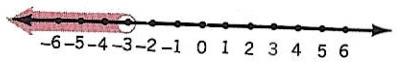
② Place your pencil point on the number line at -3. All numbers on one side of -3 make $3 - 2x > 9$ true.

Try -2. Does -2 make $3 - 2x > 9$ true? Does -1 make it true? Does 0? 1? 2?

Try -4. Does -4 make $3 - 2x > 9$ true? Does -5? Does -6? $-3\frac{1}{2}$? -3.01? -3.00001? Any number whose coordinate is to the left of -3 makes $3 - 2x > 9$ true.

③ The solution set of $3 - 2x > 9$ is $\{x : x < -3\}$.

④ The graph of $3 - 2x < 9$ is a **half-line**, the portion of a line on one side of a particular point, not including the point.



$\{x : x < -3\}$

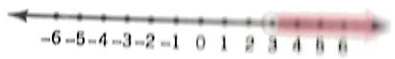
Checkpoint

1. Explain what an inequality is.
2. Explain a method for solving an inequality.
3. Describe the graph of an inequality when the replacement set is the set of integers; when the replacement set is the set of real numbers.

Exercises

A **1.** What is the solution set of $3x + 7 = 16$ if the replacement set is the set of real numbers? Which graph at the right is the graph of the solution set of $3x + 7 < 16$? of $3x + 7 > 16$?

A



B

— Solve, using the inequality postulates.

23. $5x > 45$ **24.** $-5x > 45$ **25.** $x + 9 < 16$

26. $x - 15 > -22$ **27.** $-9x < 36$ **28.** $-9x < -36$

29. $2x + 5 > 10$ **30.** $2x - 5 > 10$ **31.** $-2x + 5 > 10$

32. $-2x - 5 > 10$ **33.** $-2x + 5 < 10$ **34.** $-2x - 5 < 10$

B **35.** $12x + 3 < 3x - 9$ **36.** $-4x + 2 < x + 9$

37. $3(x + 2) < -x + 4$ **38.** $-3(x - 2) < 19$

39. $-7(-3x + 4) < 0$ **40.** $3(x + 2) > -2(x + 1)$

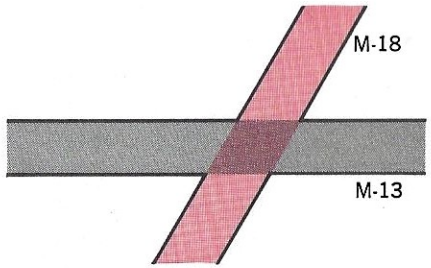
41. I am thinking of a certain set of numbers. If you multiply any one of them by 3, the result is greater than 12. Of what set of numbers am I thinking? (*Hint:* Let x represent any of the numbers in the set. Write an inequality.)

42. If 7 is added to 4 times any number in a certain set, the result is always less than 19. What is the set of numbers?

3–8 Intersection and Union of Sets

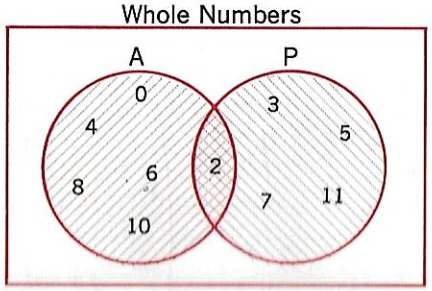
The following examples illustrate the meaning of the intersection of two sets.

EXAMPLE 1. The drawing represents two roads. How do you describe the part of the road that is a part of M–13 *and also* a part of M–18? The part of the road in common is the **intersection** of the two roads.

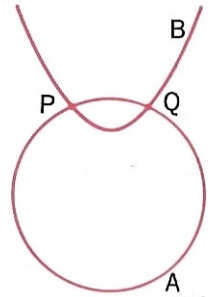


EXAMPLE 2. Set A consists of the even numbers less than 12 and set P consists of the prime numbers less than 12. What number is in A *and also* in P? The set {2} is the intersection of the two sets.

The diagram at the right is a *Venn diagram*. The points inside the rectangular region represent the whole numbers (nonnegative integers). One circular region represents set A; the other represents P. The crosshatched region represents the intersection of A and P.

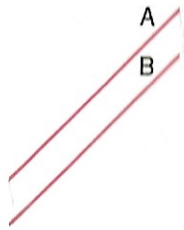


EXAMPLE 3. The drawing represents two geometric figures. Set A is a circle and set B is a parabola. Which are the points that are in A *and also* in B? The set of points that are in A and B is the intersection of the sets A *and* B. The intersection is $\{P, Q\}$.



Definition The intersection of sets A and B is a set that contains elements that are both in A *and* in B. The symbol $A \cap B$ means the intersection of A and B and is read "A intersect B."

The two lines at the right are parallel and have no points in common. Their intersection is the null set, $\emptyset$ or $\{\ \}$. Sets that have no elements in common are **disjoint sets.**



Look again at the drawing in Example 1. Think of both roads together — the road that is M–18 *or* M–13. Now you are thinking of the **union** of the two roads.

In Example 2, think of the set containing all the numbers in either of sets A *or* P. You are thinking of the set {0, 2, 3, 4, 5, 6, 7, 8, 10, 11}. Call this set C. Set C is the *union* of set A and set P. What part of the Venn diagram represents the union of sets A and P?

The set that contains all the points shown in Example 3 is the set that is the *union* of sets A and B. The union includes all the points that are on the circle and all the points that are on the parabola.

Definition The union of two sets A and B is a set C that contains all elements in either of the sets A *or* B. Elements that are common to both A and B need to be listed only once in set C. The symbol $A \cup B$ means the union of A and B and is read "A union B."

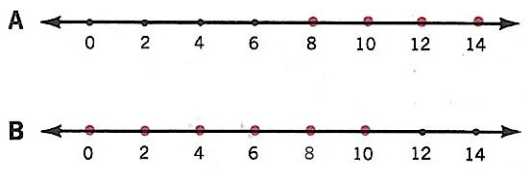
It may help you to remember the symbol $\cup$ by thinking of it as the first letter of the word *union*.

The operations of union and intersection can be applied to sets defined by sentences.

EXAMPLE 4. If the replacement set for x is $\{0, 2, 4, 6, 8, 10, 12, 14\}$ and $A = \{x : x > 6\}$ and $B = \{x : x < 12\}$, what is $A \cap B$? $A \cup B$?

$$\begin{aligned} A \cap B &= \{x : x > 6\} \cap \{x : x < 12\} \\ &= \{8, 10, 12, 14\} \cap \{0, 2, 4, 6, 8, 10\} \\ &= \{8, 10\} \\ A \cup B &= \{8, 10, 12, 14\} \cup \{0, 2, 4, 6, 8, 10\} \\ &= \{0, 2, 4, 6, 8, 10, 12, 14\} \end{aligned}$$

The number line can be used to show the graphs of $A \cap B$ and $A \cup B$. A is the graph of set A and B is the graph of set B. The points that occur in the same position on A *and also* on B are the graph of $A \cap B$. If you put the graphs over one another, the graph of the intersection is the set of overlapping points. How many points are in the graph of $A \cap B$?



The points on A *or* on B are the graph of $A \cup B$. How many points are in the graph of $A \cup B$?

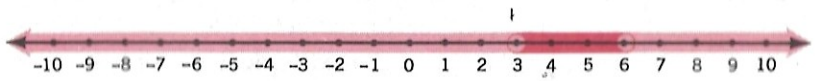
Suppose the replacement set is the set of real numbers. Use the number line to describe the graphs of the following sets.

EXAMPLE 5. If $A = \{x : x > 3\}$ and $B = \{x : x < 6\}$, graph $A \cap B$ and $A \cup B$.

The graph of set A is all the points to the right of the point for 3. The graph is a half-line.

The graph of set B is all the points to the left of the point for 6.

$A \cap B$ is the set of points that are common to both. The graph is an **open interval** (a line segment with the endpoints missing).



$A \cup B$ is the set of points that are graphs of A *or* B. It is the entire line.

Now try these

If $A = \{2, 3, 5, 7\}$, $B = \{1, 5, 7, 8\}$, and $C = \{3, 6, 9, 12\}$, what sets are named below?

1. $A \cap B$ **2.** $A \cup B$ **3.** $A \cap C$

4. $B \cap C$ **5.** $A \cup C$ **6.** $B \cup C$

Answers: **1.** $\{5, 7\}$ **2.** $\{1, 2, 3, 5, 7, 8\}$ **3.** $\{3\}$ **4.** $\emptyset$ **5.** $\{2, 3, 5, 6, 7, 9, 12\}$ **6.** $\{1, 3, 5, 6, 7, 8, 9, 12\}$

Checkpoint

1. What is the meaning of *intersection of two sets*?

2. What is the meaning of *union of two sets*?

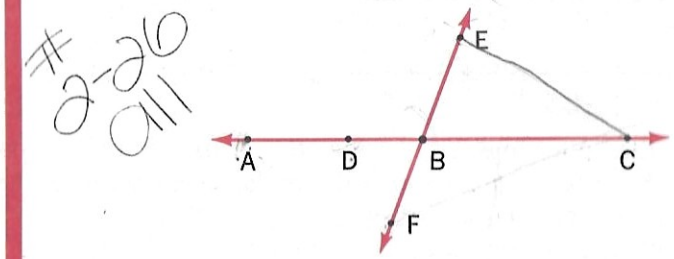
3. Is the word *and* used to define intersection or union? For which is the word *or* used?

4. How do you graph the union of two sets? the intersection of two sets?

Exercises

A **1.** If $E = \{2, 4, 6, 8, \cdots\}$ and $P = \{x : x \text{ is a prime number}\}$, find $E \cap P$ and $E \cup P$.

Use the diagram below to complete each sentence so that it is true. (Recall that $\overline{AC}$ is read "line segment AC.")



2. $\overline{AC} \cap \overline{FE} =$?

3. $\overline{AC} \cup \overline{FE} =$?

4. $\overline{BE} \cup \overline{BF} =$?

5. $\overline{AB} \cup \overline{BC} =$?

6. $\overline{EC} \cap F =$?

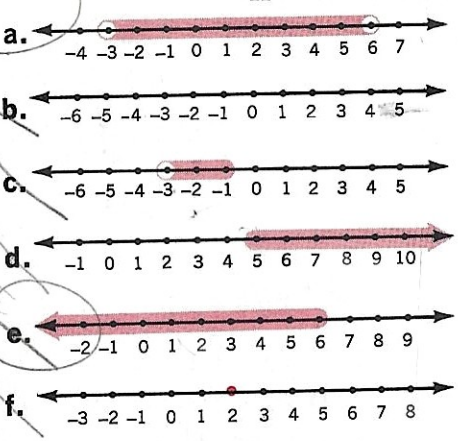
7. $\overline{AB} \cap \overline{BC} =$?

8. $\overline{BC} \cap \overline{AD} =$?

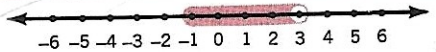
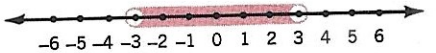
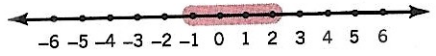
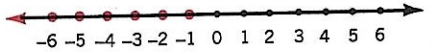
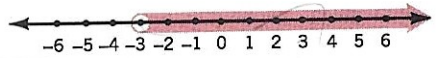
9. $\overline{AC} \cap \overline{AD} =$?

— For each of the sentences in column I, select its appropriate graph in column II. The replacement set is the set of real numbers.

I	II
10. $\{x : 3x = 15\} \cup \{x : x > 5\}$	**a.**
11. $\{x : x < 6\} \cup \{x = 6\}$	**b.**
12. $\{t : t > 4\} \cap \{t : t < -8\}$	**c.**
13. $\{t : -3 < t\} \cap \{t : t < 6\}$	**d.**
14. $\{y : y \geq 2\} \cap \{y = 2\}$	**e.**
15. $\{b : -3 < b\} \cap \{b : b \leq -1\}$	**f.**



— Use $\cup$ and $\cap$ to write names for the sets that are graphed below, as in Exercises 10–15.

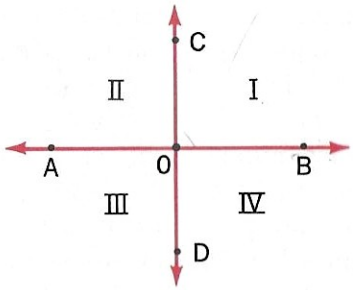
	Graph	Replacement set
16.		Real numbers.
17.		Real numbers.
18.		Real numbers.
19.		Integers.
20.		Real numbers.

— Graph the following sets. Show the graph in red or use heavier black marks. In each case the replacement set is the set of real numbers. Name each graph.

21. $\{x : x > 2\} \cap \{x : x < 5\}$
22. $\{x : x = 4\} \cup \{x : x < 3\}$
23. $\{x : x > -2\} \cup \{x : x = 2\}$
24. $\{x : x < -1\} \cup \{x : x = -1\}$
25. $\{x : x + 5 = 6\} \cap \{x : x + 3 = -8\}$
26. $\{x : 3x + 1 > 4\} \cup \{x : 2x + 3 < 15\}$

— Subsets of points in the plane are named as follows. (Recall that $\overleftrightarrow{AB}$ is read "line AB"; $\overrightarrow{OB}$ is read "ray OB" and means point O and all the points of the line on the same side as B.)

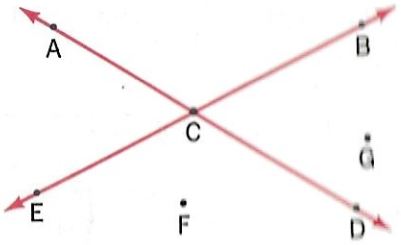
I = The set of points above $\overleftrightarrow{AB}$ and to the right of $\overleftrightarrow{CD}$.
II = The set of points above $\overleftrightarrow{AB}$ and to the left of $\overleftrightarrow{CD}$.
III = The set of points below $\overleftrightarrow{AB}$ and to the left of $\overleftrightarrow{CD}$.
IV = The set of points below $\overleftrightarrow{AB}$ and to the right of $\overleftrightarrow{CD}$.
P = The set of points above $\overleftrightarrow{AB}$.
N = The set of points to the right of $\overleftrightarrow{CD}$.



Write T or F for each sentence.

27. $\overrightarrow{OB} \subseteq$ I
28. I $\cap$ II $= \emptyset$
29. $O = (\overline{AB} \cap \overline{CD})$
30. $O \in \overline{AB}$
31. $D \in \overline{AB}$
32. P $\cap$ N $=$ I
33. (I $\cup \overrightarrow{OB}$) $\cup$ IV $=$ N

— Write *True* or *False* for each sentence, using the geometric figure at the right. Think of all the points on lines AD and EB as U, the replacement set.



34. $C = \overline{AC} \cap \overline{CD}$
35. $\overline{EB} = \overline{EC} \cup \overline{CB}$
36. $G \in$ U
37. $G \in (\overline{BC} \cup \overline{CD})$
38. $\overline{CD} \subseteq (\overline{AC} \cap \overline{AD})$
39. $\overline{EC} \cup \overline{CD} = \triangle ECD$
40. $(\overline{AC} \cup \overline{CB}) \cup \overline{AB} = \triangle ABC$
41. $\overline{AB} \cap \overline{ED} = \emptyset$
42. $F \notin \triangle ABC$

43. Is it true that A $\cup \emptyset =$ A for any set A? Explain.
44. Is it true that A $\cap \emptyset = \emptyset$ for any set A? Explain.

3–9 Translating Words into Algebraic Expressions

To solve verbal problems, you need to make certain that you can translate word expressions into algebraic symbols. An unknown number in a verbal problem may be represented by a variable. The algebraic expression involving the variable, numbers, and operations stated in a verbal expression is a mathematical model of the verbal expression. For example, a mathematical model of "Twice an unknown number subtracted from 7" would be $7 - 2n$. Notice that you must read the verbal expression carefully to make sure that the operations are performed in the correct order. Numbers in the algebraic expression will not always occur in the order that they occur in the verbal expression.

Now try these

— Which of the algebraic expressions conveys the same information as the given verbal expression?

1. The sum of an unknown number and 3.

a. $n + 3$ **b.** $n - 3$ **c.** $3n$ **d.** $\frac{n}{3}$

2. Eight less than six times some unknown number.

a. $8 - 6n$ **b.** $6n - 8$ **c.** $(6 - 8)n$ **d.** $(8 - 6)n$

3. Five times the difference when a number is decreased by six.

a. $5n - 6$ **b.** $6 - 5n$ **c.** $(5 - 6)n$ **d.** $5(n - 6)$

4. The quotient when some number is divided by 8.

a. $\frac{n}{8}$ **b.** $\frac{8}{n}$ **c.** $8n$ **d.** $8 \div n$

Answers: **1.** a **2.** b **3.** d **4.** a

Exercises

A — Use algebraic symbols to write a mathematical model of each word expression in Exercises 1–10. Choose n to represent the unknown number.

1. The sum of five times an unknown number and seven.

2. The sum of twice a certain number and twenty-seven.

3. Twice the sum of a certain number and twenty-seven.

4. The quotient formed when some number is divided by five.

5. Five less than the quotient produced when some number is divided by nine.

6. Some number is divided by thirteen and the quotient increased by eleven.

7. The sum of seventeen and the quotient formed when a certain number is divided by eight.

8. Some number decreased by nine.

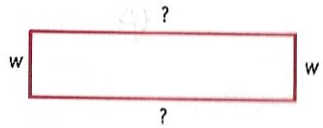
9. Twice the sum of some number and six.

10. Five times the difference formed by twelve minus some number.

11. One number is five more than seven times a second number. If you let n represent the second number, how may you represent the first number using n?

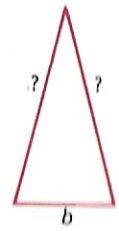
12. One number is three times a second number. To avoid fractions, let n represent the second number. Write the first number using n. Write the sum of the two numbers. Write the difference formed by subtracting the second number from the first.

13. The length of the rectangle shown here is 4 times its width, w. How will you represent its length using w? Write an expression for its perimeter in terms of w.



14. The first of two numbers is 57 more than the second. If n is the second number, what is the first in terms of n?

15. Each of the two equal sides in the triangle shown at the right is twice the third side, b. Write an expression for the perimeter in terms of b.



16. Express the sum of 3 times a certain number n and 8 times the same number.

17. The reading on Ted's bike odometer shows a mileage 6 times as much as that on Mike's odometer. Which of the two readings would you represent by n? Write an expression for the sum of the two odometer readings.

18. Sue is 8 years older than her sister Alice. Write an expression for their total ages.

19. If the difference between two whole numbers is 12 and the smaller number is n, what is the larger in terms of n?

20. If the difference between two whole numbers is 12 and the larger number is n, what is the smaller in terms of n?

21. If one addend in a sum of 36 is n, what is the other addend in terms of n?

22. If the length of a rectangle is 3 inches more than the width and the width is w, what is the perimeter?

23. How far will an airplane travel if it goes n mph for 5 hours? $2n$ mph for 6 hours?

24. To travel 500 miles at an average speed of 50 mph will take how many hours? at r mph?

25. If a is an integer, what is the next consecutive integer? (*Hint:* 24 and 25 are consecutive integers, but 32 and 34 are not.)

26. If x is an odd integer, what is the next consecutive odd integer? (*Hint:* The pairs (1, 3), (3, 5), (5, 7), (7, 9), $\cdots$, are pairs of consecutive odd integers.)

27. If y is an even integer, what is the next even integer after y?

Write an equation for each of the following statements.

28. The sum of n and $n + 57$ is 191.

29. The sum of $3n$ and $8n$ is 216.

30. The sum of $6n$ and n is 189.

31. The sum of n and $n + 8$ is 24.

32. Decreasing three times a number by 6 gives 9.

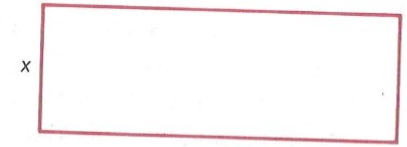
33. Eight less than 6 times some number is 10.

34. The sum of 15 and the quotient of a certain number divided by 4 is 17.

35. The difference between $4n$ and n is 29.

3–10 Conditions in Problems

Suppose that you know only one thing about the rectangle pictured at the right.



1. The length is three units more than twice the width.

From this one condition, there are an infinite number of possible dimensions for the rectangle. Which pairs of measures in the set shown below are possible dimensions for this one condition?

{1 by 5, 4 by 11, 7 by 17, $\frac{1}{2}$ by 4, 20 by 40}

Suppose you found out a second condition about the rectangle.

2. *The perimeter is 30 units.*

If condition 2 is satisfied, which of the pairs of numbers in the following set could be the dimensions?

{1 by 5, 7 by 8, 3 by 9, 4 by 11, 9 by 21}

Are there any pairs of numbers that satisfy both conditions? What is the intersection of the two sets shown above? Do the dimensions 4 by 11 satisfy both conditions of the problem? Will any other dimensions satisfy both conditions?

Consider another example. Suppose your friend is thinking about a pair of numbers and he tells you one thing about the numbers.

1. One number is five times the other.

Which of the pairs of numbers in the following set could he be thinking of?

{(1, 5), (2, 10), (−5, −25), (−5, 25), (10, 25)}

Give some other pairs of numbers that he could be thinking of.

Which of the possible second conditions below would pick out a single pair from those that are possible when using condition 1 only?

a. The sum of the numbers is even.

b. One number is $\frac{1}{5}$ of the other.

c. One number is prime.

d. The sum of the numbers is −30.

Do you see that conditions **a**–**c** do not contain enough information to tell you the unique pair? Condition **d** picks out the pair when combined with condition 1. What numbers is your friend thinking of?

Exercises

A — In each exercise below, a question is asked. Then one condition is stated as I. Which conditions in II are sufficient when combined with the information in I to answer the question?

1. What are the dimensions of the rectangle?

I. The length is 3 times the width.

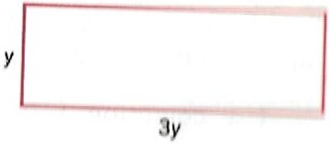
II. **a.** The length and width are integers.

b. The length is 12.

c. The width is $\frac{1}{3}$ the length.

d. The perimeter is greater than 30.

e. The length and width are prime numbers.



I. Two numbers are consecutive integers.
II. **a.** Both are even. **b.** Both are odd.
c. Both are prime. **d.** Both are composite.
e. Their sum is 20.

3. How many dollars does Neal have?
I. He has twice as many as Jody.
II. **a.** Together they have 30 dollars.
b. Jody has an even number of dollars.
c. Neal has a prime number of dollars.
d. Twice Jody's plus Neal's makes 40 dollars.
e. Jody has 10 dollars.

B **4.** Of what two numbers am I thinking?
I. Two numbers are consecutive integers.
II. **a.** Their product is odd.
b. Their sum is odd.
c. Both are prime.
d. Both are composite.
e. The square of one is equal to the other.

3–11 Solving Problems

The problems in the following examples each contain two conditions. As you know, one condition is often not enough to answer the question asked in the problem. One way to solve a problem is to use one condition to write an algebraic expression for each quantity. Then you can use the second condition to write an equation.

EXAMPLE 1. Kevin is thinking of two numbers. **The larger is five more than twice the smaller** (condition 1). *Their sum is forty-seven* (condition 2). What are the numbers?

Read the problem and ask, "What are the unknowns (the two numbers) that satisfy the conditions?"

① Use condition 1 to write the algebraic expressions.	Let x equal the smaller number; $2x + 5$ equals the larger number.
② Use condition 2 to write the equation.	$x + (2x + 5) = 47$
	$x + 2x + 5 = 47$
	$3x = 42$
③ Solve the equation and find the unknowns of the problem.	$x = 14$ (smaller number)
	$2x + 5 = 2(14) + 5$
	$= 33$ (larger number)
④ Now check to see that the numbers satisfy the two conditions of the problem.	Is 5 more than twice 14 equal to 33? Is the sum of 14 and 33 equal to 47?

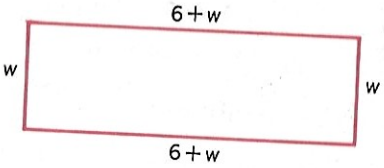
Now you try to work the same problem but use condition 2 in step ① and condition 1 in step ②. Do you get the same answer? You should.

EXAMPLE 2. **The length of a rectangle is 6 inches more than its width** (condition 1). *The perimeter is 56 inches* (condition 2). What are the dimensions?

① Unknowns are length and width.
Let w equal the width; then the length equals $w + 6$.

② $w + (w + 6) + w + (w + 6) = 56$

③ $4w + 12 = 56$
$4w = 44$
$w = 11$
Solution set: $\{11\}$



If w is 11, then $w + 6$ is 17.

④ The width is 11 inches and the length is 17 inches. Is the length 6 inches more than the width? Is the perimeter 56 inches?

Exercises

A — One condition in each problem is given in color. It will be easier to use this condition to write algebraic expressions for the unknowns (step ①). In some problems the second condition is in italics. It may be used to write an equation (step ②). Solve the following problems.

1. **The length of a rectangle is 20 feet more than its width** and *the perimeter is 240 feet.* What are the dimensions of the rectangle?

2. Gene Buckolls paid *$3612 for two cars.* **One cost $1246 more than the other.** Find the cost of each.

3. **Ellen has three times as many records as Sue.** *Together they have 32 records.* How many has each?

4. The *sum* of the angles of a triangle *is 180°*. **Angle A is twice as large as angle C, and angle B is 20° more than angle C.** How many degrees are there in each angle?

5. **Judy weighs 12 pounds more than twice the weight of her brother.** What is the weight of each if *their combined weight totals 180 pounds?*

6. **The height of the flagpole is 12 feet more than the height of Jackson High School.** *Twice the sum of the height of the school and the height of the flagpole is 140 feet.* Find the height of each.

7. Bill earns $4.50 per week on his paper route. **This is $1.00 less than twice Earl's earnings per week.** How much does Earl earn per week?

8. On a recent trip, Mr. Ridley traveled **300 miles more by plane than by train.** The distance traveled by plane was three times the distance traveled by train. How many miles did he travel in all?

9. The length of a tennis court for singles is **3 feet shorter than three times its width.** Find the length if the perimeter is 210 feet.

Number Problems

A **1.** The sum of three numbers is 72. **The first is three times the second and the third is twice the result of subtracting 6 from the second.** What are the numbers?

2. A man spent $58 for a watch, a chain, and a wallet. **He paid $2 more for the chain than for the wallet and twice as much for the watch as for the chain.** How much did he pay for each?

3. **a is 2 more than b.** The sum of 25 times a and 50 times b is 500. Find a and b.

4. **The first of two numbers is three times the second.** If 15 is added to each, the first result is twice the second result. What are the two numbers?

5. Find the **two consecutive integers** whose sum is 147.

6. There are **three consecutive integers** with a sum of 279. What are the integers?

7. There are **three consecutive integers** such that if the sum of the first two is decreased by the third, the result will be 68. What are the integers?

8. Find the **three consecutive integers** such that if three times the smallest is decreased by the sum of the other two, the difference between them will be 46.

9. Are there **two consecutive integers** such that twice the smaller plus the larger is 12? (*Hint:* If you do not get an integer when you solve the equation you write, you will know the answer is *No.*)

10. Are there **three consecutive integers** whose sum is 94? If so, find them.

11. What are the **two consecutive integers** whose sum is −25?

12. What are the **two consecutive integers** such that four times the smaller is three times the larger?

B **13.** Are there **two consecutive odd integers** whose sum is 212? If so, find them.

14. If twice the smaller of **two consecutive odd integers** is decreased by the larger, the difference is 53. What are the integers?

15. There are **three consecutive even integers** such that twice the smallest plus three times the largest will give the middle integer increased by 82. What are the 3 integers?

Age Problems

Example. **Larry's father is three times as old as Larry** (condition 1). *In 14 years, his father will be only twice as old as Larry will be at that time* (condition 2). What is the age of each now?

① The unknowns are the age of Larry now and the age of his father now.

Let Larry's age now be n;
then his father's age now is $3n$.

② To write the equation, you must state that his father's age 14 years from now equals twice Larry's age 14 years from now.

Larry's age 14 years from now equals $n + 14$,
and his father's age 14 years from now is $3n + 14$.

Hence, $3n + 14 = 2(n + 14)$ is a mathematical model of the problem.

③

	$3n + 14 = 2(n + 14)$
Use the distributive postulate.	$3n + 14 = 2n + 28$
Add $-2n$ to each side.	$n + 14 = 28$
Add -14 to each side.	$n = 14$

Solution set: {14}

If n is 14, then $3n$ is 42, so Larry is now 14 and his father is now 42.

④ Do these results check with the conditions of the problem?

B

1. Tom's present age is twice Sue's age; 10 years ago Tom was three times as old as Sue was. Find the age of each now.

2. Andy's present age exceeds Bob's age by 25 years. In 15 years Andy will be twice as old as Bob. Find their present ages.

3. Skip is now three times as old as his brother Paul. In 5 years Skip will be twice as old as Paul will be. What will be their ages in 5 years?

4. Mary is now twice as old as Jane. Seven years ago the sum of their ages was 16. What are their ages now?

5. A man is now 40 years old and his son is 14 years old. A number of years from now the father will be twice as old as his son. What is this number of years? (*Hint:* In step ①, express the age of each x years from now.)

6. Mae is now 17 years old and her aunt is 50. In how many years will Mae be one half as old as her aunt?

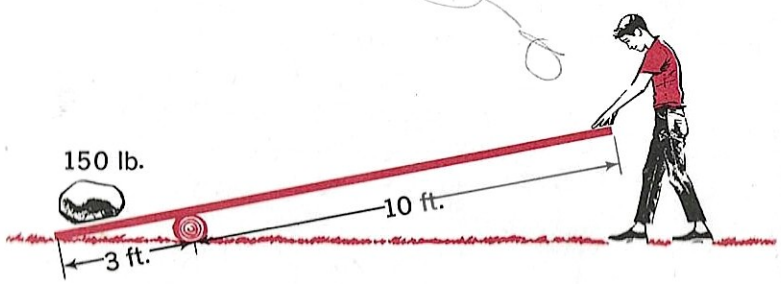
7. A father's age is 32 years and his daughter's age is 2 years. He is now sixteen times as old as his daughter. In how many years will he be only four times as old?

C

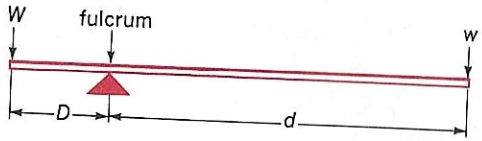
8. Jane's father is two years older than her mother. Jane's age is one third her father's age. When Jane was born, the sum of her parents' ages was 42. How old are Jane's parents now?

Lever Problems

Example. Leonard wants to move a 150-pound rock. He inserts the end of a board under the rock and then places a log under the board 3 feet from the rock. How much force (weight) must be used on the other end of the board to move the rock if the distance from the log to the other end is 10 feet?



The board acts as a lever. To solve this kind of problem, you use a **formula**, an equation that expresses a known relationship between two or more quantities. In this case you use a formula that expresses the law of the lever.



If W represents the first weight, w represents the second, D represents the distance from the fulcrum (point at which a lever is supported) to the first weight, and d represents the distance from the fulcrum to the second weight, then the formula is

$$WD = wd.$$

In this formula, the weight of the lever is ignored.

① Decide which weights and distances will be represented by which variables in the formula.

(Weight of rock)(Distance to log)
= (weight exerted)(distance to log)

② Replace the variables in the formula with numbers where possible.

$$150(3) = w \cdot 10$$

③ Solve the equation that results.

$$\begin{aligned} 150(3) &= w \cdot 10 \\ 450 &= 10w \\ w &= 45 \end{aligned}$$

Solution set: $\{45\}$

Leonard must exert 45 pounds on the log.

A

In Exercises 1–5 the board is 13 feet long and is being used to lift a rock, as in the example above.

1. If the fulcrum is 3 feet from the rock, how much weight is needed to raise a rock weighing 300 pounds? 500 pounds?

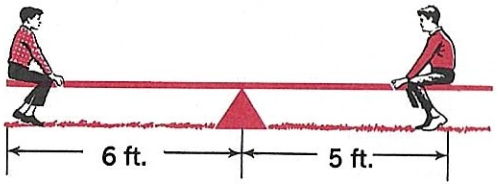
2. How much weight is needed to raise a 150-pound rock if the fulcrum is 4 feet from the rock? (The fulcrum will then be 9 feet from the other end.)

3. How much weight is needed to raise 150 pounds if the fulcrum is 7 feet from the rock?

4. How much weight is needed to raise 150 pounds if the fulcrum is 1 foot from the rock?

5. If Leonard weighs 135 pounds, what is the heaviest rock he can lift if the fulcrum is 3 feet from the rock?

6. Two boys sit on opposite sides of the fulcrum of a balanced seesaw (a lever). One boy weighs 120 pounds and sits 6 feet from the fulcrum. The heavier boy sits only 5 feet from the fulcrum. What is the weight of the heavier boy?

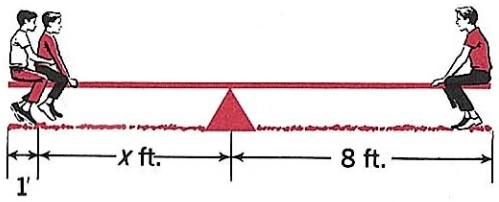


7. If a rock weighing 340 pounds is placed on a lever 2 feet to the left of a fulcrum, how far to the right must a 170-pound man put all of his weight to move the rock?

8. Two boys sit at opposite ends of a balanced seesaw. One boy weighs 20 pounds less than the other and sits 6 feet from the fulcrum. The heavier boy sits only 5 feet from the fulcrum. What is the weight of each boy?

9. Sandy weighs 92 pounds and, in order to balance a seesaw, must sit 1.5 feet farther from the fulcrum than her sister Amy, who weighs 112 pounds. How far (to the nearest tenth of a foot) from the fulcrum must each girl sit?

10. Bud and Bob balance Neal as shown in the figure. Bud weighs 96 pounds and sits 1 foot farther from the fulcrum than Bob, who weighs 88 pounds. Neal weighs 150 pounds and balances Bud and Bob by sitting 8 feet from the fulcrum. How far from the fulcrum do Bud and Bob sit?



11. Lucy Buran and her younger sister sit on opposite sides of a balanced seesaw. Lucy weighs 30 pounds more than her sister. If Lucy sits 3 feet from the fulcrum and her sister sits 4 feet from the fulcrum, find the weight of each girl.

Time-Rate-Distance Problems

Example. Two cars 400 miles apart travel toward each other. One car travels at a rate of 48 miles per hour and the other car at 32 miles per hour. In how many hours will they meet?

① Do you recognize that each car will travel the same number of hours? Then, let t equal the number of hours until the cars meet.

Then $48t$ is the distance the faster car travels in t hours and $32t$ is the distance the slower car travels in t hours.

② Do you also recognize the other condition in the problem? The distance traveled by one car plus the distance traveled by the other car is the total distance, or 400 miles.

Then $48t + 32t = 400$ is a mathematical model of the problem.

③
$$48t + 32t = 400$$
$$80t = 400$$
$$t = 5$$

Solution set: $\{5\}$

The two cars will meet in 5 hours.

④ Does the solution, 5 hours, check with the conditions of the problem?

A — In each of the following problems, you will be asked to find the rate or the time. For step ①, represent the unknown rate or time and the expressions for the distances. For step ②, write an equation involving the distance traveled. Be sure to check each rate or time with the original condition of the problem.

1. A freight train traveling at an average rate of 30 miles per hour is followed two hours later from the same station by a passenger train traveling at an average rate of 50 miles per hour. How many hours after the passenger train leaves will it pass the freight train?

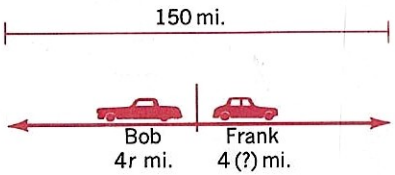
① Let t be the number of hours the passenger train travels;
then $t + 2$ is the number of hours the freight train travels;
and $50t$ is the distance traveled by the passenger train (in t hours at 50 mph),
and $\underline{?}$ is the distance traveled by the freight train (in $t + 2$ hours at 30 mph).

② Recognize that the distance traveled by the passenger train equals the distance traveled by the freight train at the point the trains pass.

$$50t = 30(t + 2)$$

Solve this equation and check.

2. Frank and Bob start from home in their cars and travel in opposite directions. Frank's rate of speed is twice Bob's. In 4 hours they are 150 miles apart. Find the rate of speed at which each travels.



① Let r be Bob's rate. Then $\frac{?}{}$ is Frank's rate, and $4r$ is the distance Bob traveled.

② $4r + \frac{?}{} = 150$

Complete the solution of the problem.

3. Two automobiles start at the same time at the same place and travel in opposite directions. One travels at the rate of 45 miles per hour and the other at the rate of 40 miles per hour. In how many hours will they be 255 miles apart?

4. Two boys on bicycles start from the same place at the same time. One rides at the rate of 8 miles an hour and the other at 5 miles an hour. They go in the same direction. In how many hours will they be 15 miles apart?

B **5.** If, in Exercise 4, the boy riding faster starts 1 hour later than the other, in how many hours will he overtake the other boy?

6. One car traveling 40 miles an hour left a certain place 4 hours later than another car traveling in the same direction at the rate of 30 miles an hour. In how many hours will the faster car overtake the other?

7. A plane leaves a certain airport at 10 o'clock, flying due north. Another plane starts from the same airport at 12 o'clock, flying due south. At 2 o'clock they are 1200 miles apart. Find the rate of speed of each, if the rate of the first is one half that of the second.

8. Mildred left camp 4 hours after Margaret did. They traveled in opposite directions. Mildred drove at 30 miles an hour, Margaret at 45 miles an hour. How many hours after Mildred left were the two girls 630 miles apart?

9. A cyclist had been traveling 15 miles an hour for 8 hours when he was overtaken by a motorist who left the same starting point 5 hours after the cyclist. Find the speed of the motorist.

10. Two drivers 240 miles apart start toward each other at the same time. The faster one travels 50 miles an hour and the slower one travels 35 miles an hour. If the faster is delayed 2 hours on the trip, how long will it be before they meet?

11. Ann starts from a certain place, traveling at 4 mph. Five hours later Fred starts from the same place and travels in the same direction at 6 mph. In how many hours will Fred overtake Ann?

12. An airplane is 75 miles directly behind a ship sailing a straight course. The rate of the plane is 120 mph and that of the ship is 20 mph. How long will it take the plane to overtake the ship?

13. At 2 P.M. a train traveling at 50 miles an hour started from Westville toward Easton. At 3 P.M. a train traveling at 40 miles an hour started from Easton toward Westville. Westville is 410 miles from Easton. At what time will the engines of the two trains pass one another? (The trains are on parallel tracks.)

14. A freight train left Kansas City for St. Louis at the rate of 12 miles an hour at the same time that a passenger train running 45 miles an hour left St. Louis for Kansas City on a parallel track. If the distance between the two cities is 285 miles, how long will it be before they pass one another?

3–12 Solving Equations for a Variable

If you were asked to solve the equation $x + a = b$ for x, you might say, "I can't solve for x until I know a number for a and a number for b." You cannot get a *numerical* solution to the equation, but you can find x in terms of a and b. The solution in terms of a and b will make the equation $x + a = b$ true when it replaces x.

Compare the solution of a specific numerical equation, shown on the left, with the solution of the general equation on the right.

EXAMPLE 1. Solve $x + 7 = 12$.

Add -7 to each side. $x = 5$

Check $5 + 7 = 12$

$12 = 12$

Solution set: $\{5\}$

Solve $x + a = b$ for x.

Add $-a$ to each side. $x = b - a$

Check $(b - a) + a = b$

$b = b$

Solution set: $\{b - a\}$

EXAMPLE 2.

Solve $38 = 16 + 2w$.

Add -16 to each side. $22 = 2w$

Multiply by $\frac{1}{2}$. $11 = w$

Check $38 = 16 + 2(11)$

$38 = 16 + 22$

$38 = 38$

Solution set: $\{11\}$

Solve $p = 2 + 2w$ for w.

Add -2 to each side. $p - 2 = 2w$

Multiply by $\frac{1}{2}$. $\frac{p-2}{2} = w$

Check $p = 2 + 2\left(\frac{p-2}{2}\right)$

$p = 2 + p - 2$

$p = p$

Solution set: $\left\{\frac{p-2}{2}\right\}$

EXAMPLE 3. Solve $-12 = -3t$.

Multiply by $-\frac{1}{3}$. $4 = t$

Check $-12 = -3(4)$

$-12 = -12$

Solution set: $\{4\}$

Solve $d = rt$ for t.

Multiply by $\frac{1}{r}$. $\frac{d}{r} = t$

Check $d = r\left(\frac{d}{r}\right)$

$d = d$

Solution set: $\left\{\frac{d}{r}\right\}$

Exercises

A — Solve the equation in column **a**. Ask yourself, "What process did I use?" Then use the same process to solve the equation in column **b** for x in terms of the other variables or numbers. Check your solutions.

	a	b
1.	$x + 5 = 7$	$x + a = c$
2.	$x - 9 = 2\frac{1}{2}$	$x - a = b$
3.	$3x = 27$	$ax = b$
4.	$\frac{x}{6} = 24$	$\frac{x}{a} = b$
5.	$5x + 4 = 10$	$ax + b = c$
6.	$\frac{3x}{5} = 2$	$\frac{ax}{b} = c$
7.	$x + 6 = 8 + \frac{1}{4}$	$x + a = b + c$
8.	$5x + 4 = 10$	$5x + a = b$
9.	$-2x = 16$	$-ax = b$
10.	$3(x + 4) = 8$	$3(x + a) = b$
11.	$-x + 4 = -5$	$-x + a = b$
12.	$-2x + 6 = 0$	$-2x + a = b$

— Solve each equation for the variable indicated.

13. Formula for the circumference of a circle; solve for d.

$$C = \pi d$$

14. Formula for the area of a parallelogram region; solve for b.

$$A = bh$$

15. Formula for a lever; solve for W.

$$WD = wd$$

16. Formula for the area of a triangular region; solve for h.

$$A = \tfrac{1}{2}bh$$

17. Formula for the perimeter of a rectangle; solve for w.

$$p = 2l + 2w$$

18. Formula for the perimeter of a triangle; solve for b.

$$p = a + b + c$$

— Solve each equation and check the results obtained.

19.	$3x + 5 + 4x = 2x + 35$	20.	$-8x - 3 + 6x = 4 + 15x$
21.	$6x - 2x = 5x + 8$	22.	$7y + 21 + y = 2y - 11 + 2$
23.	$2w - 5 = w + 8$	24.	$3t + 6 = t - 2$
25.	$3c + 4 = 9c - 4$	26.	$3 + 3v = 8 + 4v$
27.	$3x - 2 + x = x - 5x$	28.	$6 - 11h = 6h + 16$
29.	$4x + 12 = 13x - 15$	30.	$7n - 4 = 9 - 8n$

B — Solve each equation for n.

31.	$2n + a = b$	32.	$2n + 5a = 8a$
33.	$6a + 2b = 2n$	34.	$2n - a = b$
35.	$3a + n = 5b$	36.	$2n - b = 5a$
37.	$n + a = b + c$	38.	$\frac{n}{a} = b + c$
39.	$2a + 2n = b + c$	40.	$-n = -4b$
41.	$-n + a = b$	42.	$an = b + c$
43.	$-n - a = -b$	44.	$an + ac = b$

C — Solve each equation for the variable indicated.

45.	$a(b + n) = c$ for b.	46.	$\frac{n}{a} = b + n$ for a.
47.	$(a + b)n = c$ for a.	48.	$b(n - a) = c$ for n.

49. $A = \frac{1}{2}h(b + b')$ for b and also for h.

3–13 Compound Mathematical Sentences

The truth or falsity of a sentence is called the **truth value.** The following are **compound sentences** using the connective *and.* Note the truth values of the two sentences (called **components**) that make up the compound sentences and of each compound sentence. The first sentence is labeled p and the second q.

p	q
$2+3=5$ *and* $3>2$	
$2+3=5$ *and* $2>3$	
$2+3\neq 5$ *and* $3>2$	
$2+3\neq 5$ *and* $2>3$	

p	q	p *and* q
True	True	True
True	False	False
False	True	False
False	False	False

The examples illustrate the following.

The compound sentence p *and* q is true if and only if p is true and also q is true.

Compound sentences using *or* are shown below. Study the examples, which tell when the sentence p *or* q is true.

p	q
$2+3=5$ *or* $3>2$	
$2+3=5$ *or* $3<2$	
$2+3\neq 5$ *or* $2<3$	
$2+3\neq 5$ *or* $2>3$	

p	q	p *or* q
True	True	True
True	False	True
False	True	True
False	False	False

You probably have no trouble accepting the truth values of the last three sentences above. In those, the meaning of *or* is the same as in ordinary language, "one or the other, but not both." In the sentence $2+3=5$ *or* $3>2$, however, *or* is not used as it is in ordinary language. When *or* is used in mathematics, it means "either one or the other or both," unless it is otherwise stated. In fact, most legal documents use *and/or* to signify the meaning of *or* as it is used in mathematics. The examples above illustrate the following.

The compound sentence p *or* q is true if and only if p or q or both are true, that is, if and only if either component sentence is true or if both components are true.

EXAMPLE 1. If the replacement set is the set of nonnegative integers, what is the solution set of $x \leq 6$? What is the graph of $x \leq 6$?

You may think of the sentence $x \leq 6$ as the compound sentence

$$x < 6 \text{ or } x = 6.$$

The solution set for $x < 6$ is $\{0, 1, 2, 3, 4, 5\}$. Why?
The solution set for $x = 6$ is $\{6\}$. Why? Hence,
the solution set for $x \leq 6$ is $\{0, 1, 2, 3, 4, 5, 6\}$.
The graph is seven points, as shown below.

EXAMPLE 2. If the replacement set is the set of integers, what is the solution set of $-6 < x < 2$?

You may think of the sentence $-6 < x < 2$ as the compound sentence

$$-6 < x \text{ and } x < 2.$$

The solution set for $-6 < x$ is $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, \cdots\}$.
The solution set for $x < 2$ is $\{\cdots, -6, -5, -4, -3, -2, -1, 0, 1\}$.
Hence, the solution set for $-6 < x < 2$ is $\{-5, -4, -3, -2, -1, 0, 1\}$.
Explain how each solution set was obtained.

If the compound sentence uses,	then the solution set is
or	$\cup$ of solution sets of components.
and	$\cap$ of solution sets of components.

Now try these

In each of the following, determine the truth values of the components and of the compound sentence. Copy each exercise and replace each ? with either a T for True or an F for False.

1. $4 < 6$ *and* $3 < 8$

a. T **b.** ? **c.** ?

2. $-3 < 4$ *and* $0 > -1$

a. ? **b.** T **c.** ?

3. $-5 > -7$ *or* $6 > -2$

a. ? **b.** ? **c.** T

4. $3 > 2$ *or* $-5 < -6$

a. ? **b.** ? **c.** ?

5. $-3 \leq -6$ *and* $4 \geq 2$

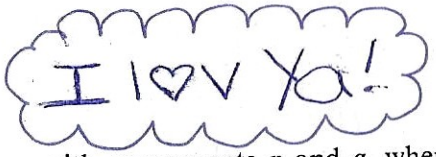
$-3 < -6$ *or* $-3 = -6$ $\quad$ $4 > 2$ *or* $4 = 2$

a. ? **b.** F **c.** T **d.** ?

e. F **f.** ?

g. ?

Answers: **1.** b. T c. T **2.** a. T c. T **3.** a. T b. T **4.** a. T b. F c. T **5.** a. F d. F f. T g. F



Checkpoint

1. In a compound sentence with components p and q, when is p *and* q true?

2. When is p *or* q true?

3. Given the solution sets of two component sentences, what is the solution set of the compound sentence whose connective is *and*? the solution set of the compound sentence whose connective is *or*?

Exercises

A — Use the roster method to show the solution sets of the following compound sentences. The replacement set is the set of integers. Graph each solution set.

1. $9 < x$ *and* $x < 12$ 10,11

2. $0 < x$ *and* $x < 5$ 1,2,3,4

3. $x = 9$ *or* $x + 3 = 10$ 9,7

4. $2x - 3 = 15$ *or* $9x = 4x + 2$ {9,3

5. $-2 < x$ *or* $x > -10$ -9,-8,...

6. $-10 < 2x$ *or* $-20 < 2x$

7. $12 \leq x$ *and* $x \leq 15$ {12,13,14,15}

8. $-10 \leq x$ *and* $x \geq -15$ {-10,-9,-8,...}

— Show that the compound sentence in **b** is equivalent to the compound sentence in **a**.

a {-5,-4,-3...2,3,4} **b** ✓

9. $2x - 3 > x - 9$ *and* $3x < 15$ {3 or $x > -6$ *and* $x < 5$

10. $-2x > 14$ *and* $9x > 45$ $x < -7$ *and* $x > 5$

11. $8 - x > 4$ *or* $3x - 9 = 16$ {...1,2,3,3 $-x > -4$ *or* $x = 8\frac{1}{3}$

12. $3x - 7 = 4(x - 2)$ *or* $9x - 3 < 32$ $x = 1$ *or* $x < 3\frac{8}{9}$

{...1,2,3,3

— For each of the following, first find a simpler compound sentence that is equivalent to the given one. Then solve the compound sentence and graph the solution set. The replacement set is the set of real numbers.

Example 1. $2x - 7 < 9$ *and* $3x - 7 > x - 9$

First find simpler equivalents to the component sentences.

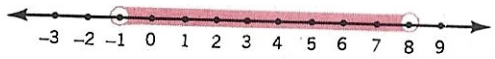
$2x - 7 < 9$ $\quad$ $3x - 7 > x - 9$

$2x < 16$ $\quad$ $3x > x - 2$

$x < 8$ $\quad$ $2x > -2$

$x > -1$

The solution set is $\{x : x < 8\} \cap \{x : x > -1\}$. The graph is shown below.



Example 2. $2x - 7 < 9$ *or* $3x - 7 > x - 9$

$2x - 7 < 9$ $\quad$ $3x - 7 > x - 9$

$x < 8$ $\quad$ $x > -1$

The solution set is $\{x : x < 8\} \cup \{x : x > -1\}$, and all real numbers are in the solution set.

13. $3x - 7 < 28$ *and* $9x - 4 > x + 4$

14. $3x - 7 < 28$ *or* $9x - 4 > x + 4$

15. $6a - 9 = 17a + 13$ *or* $3(a - 4) = 2(3a - 9)$

16. $3x - 9 > -3$ *and* $9x < 0$

— Graph each compound sentence, using the set of integers as the replacement set.

17. $12 < x < 15$

18. $0 < x < 5$

19. $-10 < x < -5$

20. $10 < 2x < 20$

21. $5 < (x + 2) < 15$

22. $0 \leq (x - 5) \leq 12$

23. The weight of each boy on the football team is more than 130 pounds and less than 205 pounds. If x represents the weight of any boy, then ? $< x <$? is true.

24. Write a compound sentence whose solution set is the set of numbers between 10 and 20.

25. Stocks are sold only in integral numbers of dollars and eighths of a dollar. The sentence $51\frac{1}{8} < x < 52\frac{3}{4}$ indicates that a stock sold on a particular day above $51\frac{1}{8}$ and below $52\frac{3}{4}$. What is the solution set for this sentence?

26. Twice the age of any student in the algebra class is more than 26, but is less than 34. If x represents the age of any pupil, then $\underline{?} < 2x < \underline{?}$ is true. Solve the inequality.

27. Billy Fitzgerald has a certain number of arrowheads. If he doubles this number and adds 4, the result is greater than 56 and is less than 60. How many arrowheads has Billy?

28. Adding the product of 417 and x to 57 gives a number that is less than $57x$ added to 2577. Find x.

B **29.** Two sides of a triangle are 3 and 7. What can you say about the length of the third side?

30. Mr. Edgar Mitchell has a cattle farm. When asked how many cows he had, he said: "Four times the number minus 167 is greater than the number plus 16; and five times the number plus 100 is less than 226 added to three times the number." How many cows has he?

3–14 Equations Involving Absolute Value

You know that if $|x|$ names 6, then x can be replaced by -6 or by 6. Then, solving $|x| = 6$ is equivalent to solving the compound sentence $x = -6$ *or* $x = 6$. The solution set is $\{-6, 6\}$.

The following examples show how you can use your knowledge of compound sentences to solve equations involving absolute value.

EXAMPLE 1. Solve $|x + 6| = 14$.

If $x + 6$ is 14 or if it is -14, $|x + 6|$ will be 14. Hence, the sentence $|x + 6| = 14$ is equivalent to the compound sentence $x + 6 = 14$ *or* $x + 6 = -14$.

$$x + 6 = 14 \text{ or } x + 6 = -14$$
$$x = 8 \text{ or } x = -20$$

Check

$|8 + 6| = 14$ | $|-20 + 6| = 14$
$|14| = 14$ | $|-14| = 14$
$14 = 14$ | $14 = 14$

Solution set: $\{8, -20\}$

EXAMPLE 2. Solve $|2k + k| - 12 = 5$.

$$|2k + k| - 12 = 5$$
$$|3k| = 17$$
$$3k = 17 \text{ or } -3k = 17$$
$$k = \tfrac{17}{3} \text{ or } k = -\tfrac{17}{3}$$

Check

$|2(\frac{17}{3}) + \frac{17}{3}| - 12 = 5$
$|\frac{34}{3} + \frac{17}{3}| - 12 = 5$
$|\frac{51}{3}| - 12 = 5$
$17 - 12 = 5$
$5 = 5$

$|2(-\frac{17}{3}) + (-\frac{17}{3})| - 12 = 5$
$|(-\frac{34}{3}) + (-\frac{17}{3})| - 12 = 5$
$|-\frac{51}{3}| - 12 = 5$
$17 - 12 = 5$
$5 = 5$

Solution set: $\{\frac{17}{3}, -\frac{17}{3}\}$

Exercises

A — Solve each of the following equations. Check before writing the solution set.

1. $\lvert a + 3 \rvert = 2$	**2.** $\lvert \frac{1}{2}k \rvert = 6$
3. $\lvert 5y \rvert = 10$	**4.** $\lvert -2v \rvert = 2$
5. $\lvert 3k - 6 \rvert = 2$	**6.** $\lvert 2y + 6 \rvert = 0$
7. $12 = \lvert 3 - x \rvert$	**8.** $\lvert \frac{5}{6} + y \rvert = \frac{2}{3}$
9. $\lvert \frac{1}{5}(2 - k) \rvert = 3$	**10.** $\lvert y + y + y \rvert = \frac{1}{3}$
11. $\lvert t + 1 \rvert = 9$	**12.** $\lvert 7x - 5 \rvert = \frac{1}{2}$
13. $\lvert \frac{1}{4}(5t - 2) \rvert = 6$	**14.** $\lvert 3d - 2 \rvert = \frac{1}{2}$
15. $\lvert 4k + 6 + 2k \rvert = 5$	**16.** $\lvert 15 \rvert = \lvert w + 2 \rvert$
17. $\lvert \frac{2}{3}y \rvert = \frac{1}{12}$	**18.** $\lvert n \rvert + 6 = 2$
19. $\lvert 3m + 6 \rvert = 4$	**20.** $\lvert -3m \rvert = 14$
21. $6 - 2 = \lvert x \rvert + 3$	**22.** $\lvert \frac{1}{2}w + \frac{1}{2}w \rvert = 1$
23. $\lvert 2w + w \rvert - 2 = 8$	**24.** $\lvert 6r + 2 \rvert + 6 = 9$

B — Find a simple name for each set.

Example 1. $A = \{x : 2(x + 2) - 3(x + 5) = -(x + 11)\}$

$$2(x + 2) - 3(x + 5) = -(x + 11)$$
$$2x + 4 - 3x - 15 = -x - 11$$
$$2x - 3x + 4 - 15 = -x - 11$$
$$-x - 11 = -x - 11$$

The equation is true for all replacements for x. Hence, A is the same as the set of real numbers. R is the simple name for the set.

Example 2. $B = \{y : 5(y-3) + 2(y+2) = 3y + 2 + 2(2y-5)\}$
Solving the sentence gives the following.

$$7y - 11 = 7y - 8$$
$$-11 = -8$$

The equation is true for no values of y. Hence, set B is the same as $\emptyset$. $\emptyset$ is the simple name.

25. $A = \{x : x - 1 = -x - (x+2) + 3x\}$
26. $E = \{t : 4(2t-3) - 6(t-2) = 2(t+4)\}$
27. $F = \{f : -1 + 5f = 5(f - \frac{1}{5})\}$
28. $G = \left\{g : \frac{g-1}{4} = 2g + 5\right\}$
29. $H = \{a : 5 \cdot |a+2| = 25\}$
30. $I = \left\{k : \frac{(\frac{2}{3})(k)}{\frac{1}{3}} = \frac{1}{12}\right\}$
31. $J = \{n : n - 1 = n - (n-1) + n\}$
32. $K = \{c : (\frac{1}{3})(c+2) - 3(12c - \frac{1}{3}) = -3(c+4) - \frac{1}{2}(3c+6)\}$
33. $L = \{d : 3(2d-5) = 5(4-d)\}$
34. $A = \{b : 3(b-2) + 3(b-3) = b\}$
35. $Z = \{y : \frac{1}{4}y + 12 = -(y+2) + \frac{1}{4}\}$
36. $A = \{x : |x| = -3\}$
37. $C = \left\{y : y + 2 = \frac{2y+16}{-10}\right\}$
38. $B = \left\{t : \frac{(3t-5)}{6}(\frac{2}{3}) = 3t + 1\right\}$
39. $Y = \{y : 3(y-2) + 6 = 2y\}$
40. $S = \left\{t : (\frac{2}{3})(t-2) = t + 2 - \left(5 - \frac{t}{3}\right)\right\}$
41. $T = \left\{k : 3k + 1 = -k\left(\frac{1}{k} - 3\right)\right\}$
42. $A = \left\{u : \left|\frac{2+3u}{4}\right| = 5\right\}$
43. $B = \{m : |m + 2m| = \frac{1}{2}\}$
44. $D = \{d : 2|d+1| = 6\}$

More Challenging Problems

1. George rode out of town on the bus at an average speed of 20 miles per hour and walked back at an average speed of 3 miles per hour. How far did he go out of town if the entire trip took eight hours and if two hours of this was spent visiting his friend Tom?

2. A ship can go downstream from town A to town B at 20 miles per hour in five hours less time than it takes to go upstream from B to A at 15 miles per hour. How far apart are the towns?

3. At what two times between 7 o'clock and 8 o'clock will the minute hand and the hour hand of a clock be exactly 15 minute spaces apart?

4. A man bought two airplanes but found them unsatisfactory for his purpose. He sold them for \$6000 each, making 20% on one and losing 20% on the other. Did he make money or lose money and how much?

5. An Australian farmer divided his sheep among his three sons. Alfred got 20% more than John, and Alfred got 25% more than Charles. John's share was 3600. How many sheep did Charles get?

6. Mr. Teal has a 99-year lease on a piece of property. When asked how much of the lease had expired, he said: "Two thirds of the time past is equal to four fifths of the time to come." How much of the lease had expired?

CHAPTER REVIEW

1. Know the meaning and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

equivalent sentences (*102*)
addition postulate for equations (*103*)
multiplication postulate for equations (*105*)
solution, or root, of an equation (*108*)
set-builder notation (*110*)
identity (*114*)
conditional equation (*115*)
comparison postulate (*118*)
half-line (*119*)
intersection (*123*)
disjoint sets (*123*)
union (*123*)
open interval (*124*)
formula (*136*)
compound sentence (*144*)
truth value (*144*)
components (*144*)

If the replacement set is $U = \{-3, -1, 0, 1, 3\}$, what is the solution set of each sentence?

2. $2x + 3 = x$
3. $4 + 3x > 10$
4. $2y + 7 = 7 + 2y$
5. $2x = 6$ *and* $19 + x = 16$
6. $2x = 10 - x$
7. $x > x + 1$
8. $a = 4 + a$
9. $8(3 + x) = 5x + 15x$ *or* $6x = 9x$

10. Is the equation $4(x-3) - 2x = -12 + 2x$ a conditional equation or an identity? How do you know?

11. Write *Yes* if the sentence in column II is equivalent to the sentence in column I; otherwise, write *No*.

	I	II
a.	$6a + 2 - 2a = a - 1$	$3a + 2 = -1$
b.	$7x = 14$	$6x + x = -x + 14 + x$
c.	$9y + 6 = 18y$	$-9y + 4 = 0$
d.	$3x - 2 = 18x - 17$	$0 = 15x - 15$
e.	$\frac{3}{4}x = 17$	$\frac{4}{3} \cdot (\frac{3}{4}x) = 17 \cdot \frac{4}{3}$
f.	$3(n - 9) = 6 + 3n$	$3n - 9 = 6 + 3n$
g.	$-\frac{7}{6}x = 1$	$x = -\frac{6}{7}$
h.	$an = b + c$	$n = \frac{b}{a} + c$

— Find the solution set of each sentence if the replacement set is the set of real numbers.

12. $3n = 2$

13. $|-2n| = 3$

14. $\frac{n}{2} + \frac{3}{4} = 6$

15. $6 + 2x > 8$

16. $-4x > 20$

17. $9x + 2 > -7$ *or* $x + 9 < 0$

18. $|-n + 7| = -2$

19. $3y - 2y < 12$ *and* $-2y < 4$

— Write *True* if the sentence is true and *False* if it is false.

20. $\forall x \quad 4x + 2x = 6x$

21. $\forall x \quad x + 2 > x$

22. $\exists x \quad x > x + 2$

23. $\forall x \quad 3x - 9 = x + 6$

24. $\exists x \quad 2x + 5 = 17$

25. $\exists x \quad 5(x + 2) < 3(x - 9)$

— For each inequality, give the solution set and graph it on a number line. The replacement set is R.

26. $x + 4 < 8$

27. $2x - 2 > 7$

28. $\frac{1}{2}x - 3 > 1$

29. $-4x \leq 8$

30. $2(x - 5) - 2(x + 3) \leq 14$

31. $12 - 9x < -8x$

— Use the roster method to list the members of these sets if the replacement set is the set of integers.

32. $\{x : x + 15 = 29\}$

33. $\{x : x + 2 > 14\}$

34. $\left\{n : 2\frac{2}{5} - \frac{n}{10} = \frac{7n}{10}\right\}$

35. $\{y : y + \frac{1}{2} = 5\}$

36. Describe the solution set of a compound sentence whose connective word is *and*.

37. Describe the solution set of a compound sentence whose connective word is *or*.

— Find the solution set of these compound sentences and graph it. The replacement set is the set of integers.

38. $-3 > x$ *and* $x < 4$

39. $2x - 7 < x + 2$ *and* $x > -1$

40. $-3 < x$ *or* $x > 4$

41. $5x + 4 \geq 9$ *or* $3x - 2 < -8$

— Show that the compound sentence in column II is equivalent to the compound sentence in column I.

	I	II
42.	$x + 2 < 11$ *and* $x - 17 < -5$	$9 > x$ *and* $x < 12$
43.	$12 \geq x$ *and* $x \leq 15$	$x - 6 \leq 6$ *and* $x - 8 \leq 7$
44.	$x = 1$ *or* $x < 6$	$3x - 4 = 7$ *or* $\frac{1}{2}x < 3$
45.	$-2 < x$ *and* $x > -10$	$5x - 3 > -13$ *and* $x + 3 > -7$
46.	$x < -7$ *or* $x > 5$	$-2x > 14$ *or* $9x > 45$

— Solve the equation in **a**. Then use the same process to solve the equation in **b** for the same variable as in **a**.

	a	b
47.	$4n = 19$	$bn = a$
48.	$3n + 7 = 42$	$3n + b = c$
49.	$\frac{2b}{3} = 5$	$\frac{ab}{c} = f$

— Each sentence below expresses one condition of a problem. Use one variable to write algebraic expressions for the unknowns (given in parentheses) that use the condition.

Example. The sum of two numbers is 42. (the two numbers)

Let x = one number
$42 - x$ = the other number

50. Two cars are 500 miles apart. (distance each traveled)

51. The length of a rectangle is twice the width. (length and width)

52. The greater of two numbers is 10 less than twice the smaller. (two numbers)

53. Paul's 25 coins are dimes and quarters. (number of coins of each)

— Solve the following problems.

54. After an airplane has been flying west from Chicago at 150 miles per hour for 5 hours, it is passed by another airplane that left Chicago 2 hours after the first airplane left. At what speed is the second airplane traveling?

55. Are there three consecutive integers such that twice the first added to the second gives 7 more than 4 times the third? If there are, find them.

CHAPTER TEST

Find the solution set of these sentences if the replacement set is the set of real numbers.

1. $5y - 6 = 2y + 9$

2. $2(7 - x) = 5(x + 3) - 6$

3. $2(x - 6) + 3(7 - x) = 5$

4. $|-9k| < 33$

5. $2x + 3 > 7$

6. $9 - 3x < 15$

7. $3x + 2 > 11$ *and* $5x < 35$

8. $-15 < 3x < 12$

Write *True* or *False* for each statement.

9. $\forall x \quad 8 - x = 8 + (-x)$

10. $\forall a \quad \frac{1}{a} \cdot a = 1, a \neq 0$

11. $\exists x \quad -19\frac{1}{6} + x = 0$

12. $\{x : 2x - 5 = 13\} = \{9\}$

13. The equation $2x - 19 = 5x + 27$ is equivalent to the equation $3x = 46$.

14. $\{n : na + b = c\} = \{n : n = \frac{c}{a} - b\}$

15. The solution of $2x - 1 = 8 - 7x$ is -1.

16. $\{x : 3 < x < 9\} = \{x : 3 < x \text{ or } x < 9\}$

17. $\{x : x \geq 7\} = \{x : x > 7 \text{ or } x = 7\}$

Solve for the variable indicated.

18. $A = bh$, for h.

19. $P = 2l + 2w$, for w.

20. $V = dr$, for d.

21. The sum of two numbers is 57. One number is 13 more than the other. What are the numbers?

22. Are there three consecutive even integers whose sum is 42? If so, find the numbers.

23. A car traveling 50 mph leaves A for B at 8 A.M. Another car leaves A at 10 A.M. heading for B traveling 60 mph. At what time will the second car catch the first?

24. The difference between two numbers is 6. If three times the larger is subtracted from 52, the result is the same as if 56 is subtracted from seven times the smaller. Find the numbers.

25. Alfred is now twice as old as his sister. Eight years ago the sum of their ages was 11. What are their ages now?

CUMULATIVE REVIEW

Give the letter of the response that best completes the sentence or answers the question.

1. Which of the following sets is not well-defined?

a. The set of odd numbers. **b.** The set of children.
c. The set of 1923 cars. **d.** The set of electrical engineers.

2. Which set is equivalent to $\{3, 4, 5, 4\}$?

a. $\{3, 4, 5\}$ **b.** $\{3, 4, 5, 3\}$
c. $\{0, 2, 4, 6, 8\}$ **d.** $\{3, 4, 5, \cdots\}$

3. Which replacements for a, b, c, and d make $\frac{a}{b} = \frac{c}{d}$ true?

a. $a = 2, b = 3, c = 4, d = 5$
b. $a = 7, b = 6, c = 2, d = 9$
c. $a = 13, b = 16, c = 65, d = 80$
d. $a = 15, b = 45, c = 30, d = 21$

4. Which of the following sentences is false?

a. $|-6| > 0$ **b.** $-(2) = -2$ **c.** $(-1)(-2) = -(2)$ **d.** $-(-\frac{1}{2}) = \frac{1}{2}$

5. A simplified form of $\frac{-2 + 7}{-\frac{1}{2}}$ would be

a. $\frac{5}{-\frac{1}{2}}$. **b.** $-2\frac{1}{2}$. **c.** $4\frac{1}{2}$. **d.** -10.

6. Which of the following illustrates the distributive postulate for fractional numbers?

a. $\frac{1}{2} + 0 = \frac{1}{2}$ **b.** $\frac{3}{4}(12 + 1\frac{1}{3}) = (\frac{3}{4} \cdot 12) + (\frac{3}{4} \cdot 1\frac{1}{3})$
c. $\frac{5}{6} \times \frac{3}{5} = \frac{3}{5} \times \frac{5}{6}$ **d.** $\frac{3}{2} \times \frac{2}{3} = 1$

7. Which group of three measurements may be the lengths of the sides of a right triangle?

a. $x = 15, y = 8, z = 55$ **b.** $x = 20, y = 30, z = 15$
c. $x = 5, y = 7, z = \sqrt{74}$ **d.** $x = \sqrt{7}, y = \sqrt{12}, z = 21$

8. Which is a true sentence?

a. $|-7| = |0 + 11 - 4|$ **b.** $-6 < [-4 + (-3)]$
c. $[(+4) \cdot (+5)] > [(-4) \cdot (-5)]$ **d.** $(-6 - 2) = 6 + 2$

9. What is another name for $12xy + 3y - x + 2y$?

a. $16xy$ **b.** $12xy + 5y - x$ **c.** $9xy + 2y$ **d.** $12xy - x$

10. For which is +19 the correct difference?

a. $(-11) - (8)$ **b.** $-11 - (-8)$
c. $11 - (+8)$ **d.** $11 - (-8)$

11. If a is -3, b is -8, and c is -6, the value of $\frac{ab}{2c}$ is

a. 12. **b.** -12. **c.** 2. **d.** -2.

12. When three negative numbers are multiplied, the product is

a. larger than any of the numbers.
b. the product of the absolute values of the numbers.
c. the negative of the product of the absolute values of the numbers.
d. a positive number.

13. Which sentence is true if n is replaced by -6?

a. $3n = 18$ **b.** $-3(n) = -18$
c. $3n = -18$ **d.** $-4n = -24$

14. If the replacement set is the set of real numbers, which of the following is not true?

a. $\forall x \quad x = x$ **b.** $\forall x \quad x \cdot 1 = x$
c. $\forall x \quad x \cdot x = 2x$ **d.** $\forall x \quad x + 0 = x$

15. If the replacement set is the set of real numbers, what is the solution set for $-3(x + 4) = 18$?

a. $\{-2\}$ **b.** $\{-10\}$ **c.** $\{-4\frac{2}{3}\}$ **d.** $\{7\}$

16. If A = {a, b, c, d} and B = {b, c}, then A $\cap$ B is

a. {a, b, c, d}. **b.** $\emptyset$.
c. {b, c}. **d.** {a, d}.

17. Solving $\frac{3x + y}{a} = 3z$ for y gives

a. $y = 3za - 3x$. **b.** $y = 3z - \frac{3x}{a}$.
c. $y = \frac{3z}{a} - 3x$. **d.** $y = 9xza$.

18. The set of ? is dense.

a. Integers. **b.** Whole numbers.
c. Natural numbers. **d.** Real numbers.

19. Fred has \$2.25 in dimes and nickels. If he has twice as many dimes as nickels, how many of each coin does he have?

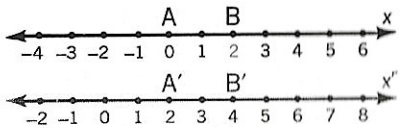
a. 20 dimes and 10 nickels
b. 18 dimes and 9 nickels
c. 16 dimes and 8 nickels
d. 14 dimes and 7 nickels

20. What rational number is equal to $\frac{11}{15}$?

a. $\frac{2}{3}$ **b.** $\frac{4}{5}$ **c.** $\frac{22}{30}$ **d.** $\frac{30}{45}$

ENRICHMENT

Linear Transformations



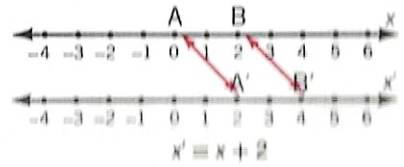
Look at the number line, x, shown in black at the right. Cover everything in the diagram except the black line. Now there is nothing to indicate where the point for 0 should be or where the point for 1 should be. You only know that 1 should be to the right of 0.

In other words, the name, or **coordinate**, of a point on a number line depends upon where 0 is placed and the distance between 0 and 1.

Uncover the diagram and look at the gray number line, x'. The distance between 0 and 1 is the same as it is on x. But 0 on x' is in the position that corresponds to -2 on x. And 2 on x' is in the position that corresponds to 0 on x. What point on x' corresponds with -1 on x? with 3 on x? with 6 on x? In general, how do the x' coordinates compare to the x coordinates? The number line is shifted two units to the left. This can be expressed as a linear equation.

$$x' = x + 2$$

If the number lines are kept in corresponding positions, then the points will be shifted. The points corresponding to A and B will be A' and B', respectively, which are two units to the right of the original points. This relocation can be shown by the diagram at the right.

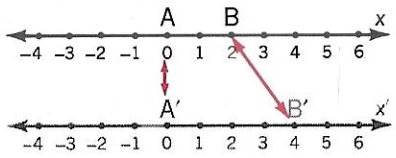


Any system of relocating points is called a **transformation of coordinates.** The two transformations above are the same; $x' = x + 2$

has just been represented in two different ways. In the rest of this section, the number lines will remain in corresponding positions and the points will be shifted.

A transformation of coordinates on the number line in such a way that the new coordinate x' can be found from the original by an equation of the form $x' = x + a$, for any real number a, is a **translation.** It shifts all coordinates to the left or to the right $|\ a\ |$ units.

Another transformation of basic importance is one such as $x' = 2x$. Here each point with coordinate x is relocated with a coordinate equal to twice the original coordinate. Thus, the point originally labeled 0 is still 0; 1 is now 2; 2 is now 4; and so forth. This is the **stretching transformation.**

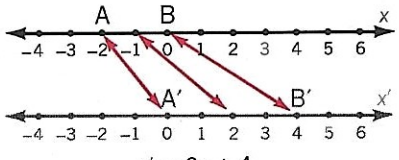


$x' = 2x$

What do transformations do to measures of lengths? The first transformation, $x' = x + 2$, is a translation. Point A is labeled 0, and A after the translation, A', is labeled 2. Point B is labeled 2, and B' is labeled 4. Then the length of $\overline{AB}$, symbolized $|\ \overline{AB}\ |$, is $|\ 2 - 0\ |$ or $|\ 0 - 2\ |$, or 2; $|\ \overline{A'B'}\ |$ is $|\ 4 - 2\ |$, or 2. The measure of the length of the segment was not changed by translation.

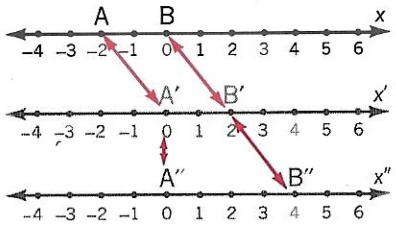
The second transformation is $x' = 2x$, stretching. Again, let A be 0 and B be 2. Then A' is 0 and B' is 4. Now $|\ \overline{AB}\ |$ is $|\ 2 - 0\ |$, or 2, but $|\ \overline{A'B'}\ |$ is $|\ 4 - 0\ |$, or 4. The measure of the length is double.

One of the most important transformations used in mathematics is the **linear transformation,** defined by $x' = ax + b\ (a \neq 0)$. The number lines illustrate the linear transformation $x' = 2x + 4$.



$x' = 2x + 4$

This transformation is frequently shown by two successive transformations, and the end result is called the **product** of the two transformations. Suppose the coordinates of A and B are transformed by $x' = x + 2$ and then by $x'' = 2x'$. The end result is shown by the third line, which is the same as the line for $x' = 2x + 4$.



$x' = x + 2 \qquad x'' = 2x'$

Algebraic results are the same; $x + 2$ can be substituted for x' in $x'' = 2x'$ and the transformation is $x'' = 2(x + 2)$, or $x'' = 2x + 4$.

Exercises

1. Let A be -1 and B be 2. Draw number lines to illustrate the following transformations. The new points will be A' and B'.

a. $x' = \frac{1}{2}x$ **b.** $x' = x + 3$ **c.** $x' = 3x + 2$

2. Show by drawing number lines that the following two transformations are equivalent.

a. $x' = x + 1$ followed by $x'' = 3x'$. **b.** $x'' = 3x + 3$

3. For each of the following, determine a translation and a stretching whose product is the linear transformation.

a. $x'' = 2x + 4$ **b.** $x'' = \frac{1}{3}x + 3$ **c.** $x'' = x + 5$

4. Pick a particular case and show that a linear transformation $x' = ax + b$, $a \neq 0$, $b \neq 0$, multiplies the measure of the length of a segment by $|\ a\ |$.

5. Determine the linear transformation that relabels $X(1)$ as $X'(5)$ and $Y(2)$ as $Y'(7)$.

6. Show that if a translation leaves the coordinate of any point unchanged, it leaves them all unchanged. (*Hint:* Show a is 0.)

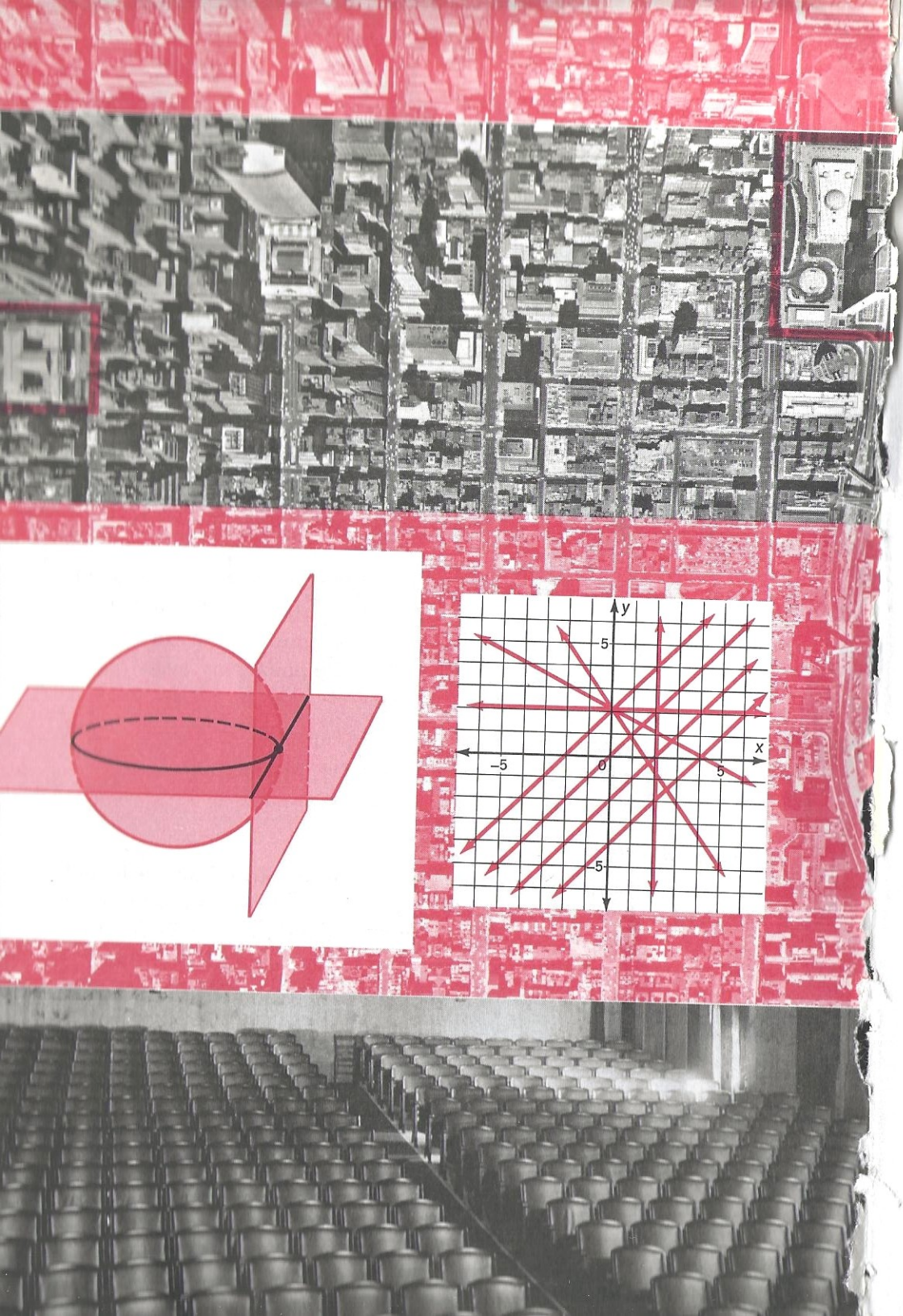
7. Show that if a stretching leaves the coordinate of any point, except 0, unchanged, then it leaves all the points unchanged.

THE GROWTH OF MATHEMATICS

Egyptian Algebra

Much of Egyptian arithmetic and algebra is known from the Rhind Papyrus of about 1700 B.C. It is a collection of 85 problems with answers (but little explanation), copied by the scribe Ahmes from an older manuscript, and called "directions for knowing all dark things."

The first part of the work deals with fractions because they presented such problems to the Egyptians. They could only work with fractions with numerators of 1. The next part deals with the fundamental operations of arithmetic. Then simple algebraic problems are treated. One problem is to find a number such that "the sum of it and one seventh of it shall together equal 19." The answer is $16\frac{8}{8}$, but it was given in an interesting form: $16 + \frac{1}{2} + \frac{1}{8}$. Symbols were used for an unknown and to show addition, subtraction, and equality. But the Egyptians did not attain in algebra what they did in geometry.



CHAPTER 4

GRAPHING RELATIONS

Many ideas of geometry are included in this chapter. You may be reviewing some terms you already know or you may be learning them for the first time.

● The figures in the center left diagram illustrate many geometric figures. The vertical plane and the sphere intersect in a point. What is the intersection of the horizontal plane and the sphere? of the two planes? How does the model of a plane differ from a geometric plane? How does the intersection of the two planes differ from a line?

● Look at the diagram at the center right. Several of the lines have a point in common. Several of the lines are parallel; they will never intersect. Yet they have something in common. What is it? What kind of lines are the ones labeled x and y? Can you tell which of the lines is defined by $y = -\frac{1}{2}x + 2$ just by looking at the equation?

● Suppose a UN delegate in New York City wanted to walk from the corner of 46th Street and First Avenue to the Public Library at 42nd Street and Fifth Avenue. How many different ways are there for him to reach the Library? If he wanted to tell someone else how to get to the Library, what would be the simplest directions he could give? Do you reach the same spot by taking a zigzag path as you would if you went in a straight path?

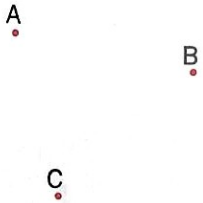
● Usually, seats in a theater are located by two numbers — row and seat. If you bought a theater ticket with only a row number on it, would you know exactly where to sit? Would it be enough just to have a seat number? What would tell you *exactly* where to sit? Would it make any difference if (3, 11) meant "row 3, seat 11" or "seat 3, row 11"? What would be the set of chairs with the same seat number as row number? Are any two seat numbers or row numbers the same?

4–1 Points, Lines, and Planes

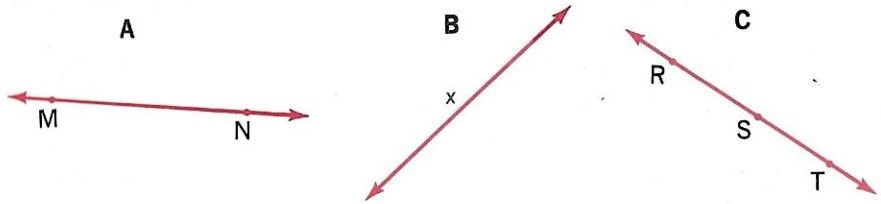
From your previous study of geometry, can you tell what each of the following is: Point, line, line segment, ray, half-line, plane, half-plane, parallel lines, and perpendicular lines?

In geometry, *point*, *line*, and *plane* are **undefined terms.** They are the basic concepts needed to define the other terms. Although not formally defined, they can be described to clarify their meaning.

A **point** shows position. If you thought of a dot on a table becoming smaller and smaller until it vanished, the spot where it vanished should make you think of a point. It has no length, no width, no thickness. Points are *represented* by dots and named by capital letters.

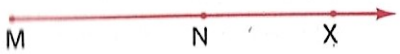


A **line** is an infinite set of points. If you thought first of two points and then of all the points on the straight path between them and of all the points extending without bound in either direction, you would be thinking of a line. A line has no width or thickness. A line is named by any two points on it or by one lower-case letter. The line shown in A below is named $\overleftrightarrow{MN}$. Give names for the lines shown in B and C. The points on a line are *collinear* points.



A **line segment** is a part of a line. In A above, points M and N and all the points between them make up a line segment. A segment, too, is an infinite set of points because there is an infinite number of points between any two points. Recall that a segment is named by a bar over the two letters, $\overline{MN}$, rather than the arrow. Points M and N are the **endpoints** of the segment.

Think of $\overline{MN}$ extended to the right to become $\overline{MX}$, as shown below. Now think of the segment extending to the right without bound, so far that for any boundary you could think of in that direction, the line would pass through it. You are thinking of a **ray.**

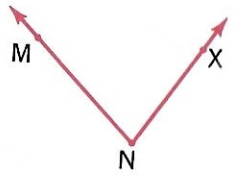


Definition A ray is a set of points formed by extending a line segment without bound in one direction only.

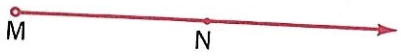
The ray at the bottom of the previous page is made up of point M and all the points on the N-side of M. The ray is named $\overrightarrow{MN}$. Point N and all the points on the M-side of N is ray NM, or $\overrightarrow{NM}$.

Another way to describe a line is to say that it is the union of two collinear rays such as $\overrightarrow{MN}$ and $\overrightarrow{NM}$. When you see the word *line*, it always means "straight line."

The union of two *noncollinear* rays with a common endpoint forms an **angle** such as angle MNX. The point where the two rays meet is called the **vertex.**

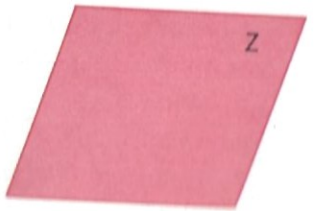


Think of ray MN. Now think of the same set of points without point M. You are thinking of a half-line. A **half-line** is a ray without the endpoint of the ray. The half-line shown below would be named $\overleftarrow{MN}$. The circle indicates that point M is the boundary of the half-line but not part of it.

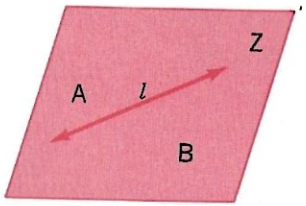


Why is the union of $\overrightarrow{MN}$ and $\overrightarrow{MX}$ not $\overleftrightarrow{XN}$, shown below?

A **plane** is an infinite set of points extending without bound in all directions. Any perfectly flat surface you can think of *represents* a *part* of a plane. Can you think of anything that would truly represent a plane? A plane has no thickness and no edges anywhere. A plane, Z, can be represented as shown at the right.

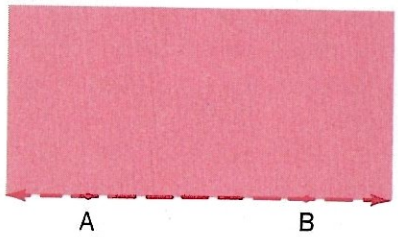


Think of line l contained in plane Z. Plane Z is separated into three sets: the set of points on the A-side of l, the set of points on the B-side of l, and the set of points that is l. The sets of points on either side of l are called **half-planes.**



Definition A half-plane is the set of points of a plane that is on one side of a line.

The line is the **boundary,** or edge, of each half-plane, but its points do not belong to either half-plane. A half-plane can be shown with a dotted line for the boundary.



A **polygon** is a simple closed path formed by the union of three or more line segments. A polygon separates a plane into three sets of points: the points of the polygon, the points in the **interior** of the polygon, and the points in the **exterior** of the polygon.

Two lines in a plane that have no points in common are **parallel lines.** Two lines that intersect (have one point in common) forming right angles (angles of 90°) are **perpendicular lines.**

Checkpoint

1. Can you *define* a point? Can you give an example that helps you think of a point?

2. What is the relationship between a point and a line? a ray and a half-line? a plane and a half-plane?

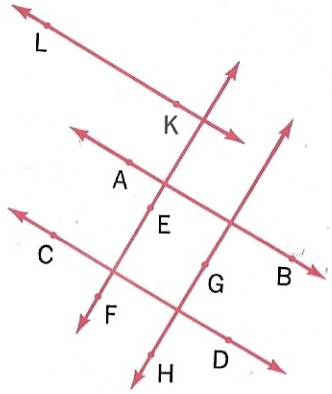
3. Define *parallel lines; perpendicular lines.*

4. Define *without bound.*

Exercises

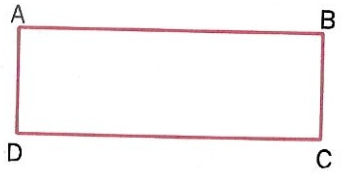
A

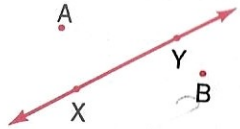
1. Name three pairs of parallel lines and three pairs of perpendicular lines in the drawing at the right.



2. Two segments are parallel if they are parts of parallel lines. How would you define *perpendicular segments*? *perpendicular rays*?

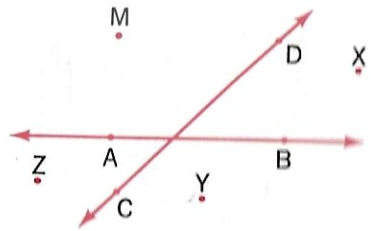
3. Name four pairs of perpendicular segments in rectangle $ABCD$. Name two pairs of parallel segments.





4. Think of any two points in the plane on the A-side of $\overleftrightarrow{XY}$. Does any segment with the two points as endpoints contain a point of $\overleftrightarrow{XY}$? Think of a point on the A-side and a point on the B-side. Does the segment with the two points as endpoints contain a point of $\overleftrightarrow{XY}$?

5. Draw a separate diagram to show the points on the



a. D-side of $\overleftrightarrow{AB}$.

b. C-side of $\overleftrightarrow{AB}$.

c. A-side of $\overleftrightarrow{DC}$.

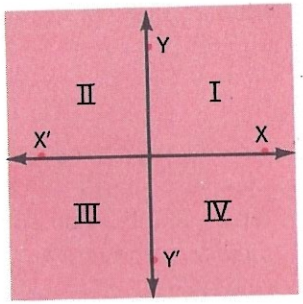
d. Y-side of $\overleftrightarrow{CD}$.

e. D-side of $\overleftrightarrow{AB}$ and also on the B-side of $\overleftrightarrow{DC}$.

f. C-side of $\overleftrightarrow{AB}$ and also on the A-side of $\overleftrightarrow{CD}$.

g. M-side of $\overleftrightarrow{CD}$ and also on the Y-side of $\overleftrightarrow{CD}$.

6. The drawing represents two perpendicular lines, $\overleftrightarrow{YY'}$ and $\overleftrightarrow{XX'}$, and all the points in a plane. The four **quadrants** bounded by the lines are named I, II, III, and IV. Which quadrant contains



a. X-side of $\overleftrightarrow{YY'} \cap$ Y-side of $\overleftrightarrow{XX'}$?

b. Y'-side of $\overleftrightarrow{XX'} \cap$ X-side of $\overleftrightarrow{YY'}$?

c. X'-side of $\overleftrightarrow{YY'} \cap$ Y'-side of $\overleftrightarrow{XX'}$?

d. X'-side of $\overleftrightarrow{YY'} \cap$ Y-side of $\overleftrightarrow{XX'}$?

7. Use the drawing for Exercise 6. Which points are

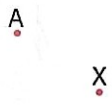
a. on neither side of $\overleftrightarrow{YY'}$? on neither side of $\overleftrightarrow{XX'}$?

b. not in I, II, III, or IV?

c. on neither side of either line?

d. the union of I, II, III, IV, $\overleftrightarrow{YY'}$, and $\overleftrightarrow{XX'}$?

8. Consider any three points A, M, and X not on one line. Make a separate drawing to represent the points for each of the following.



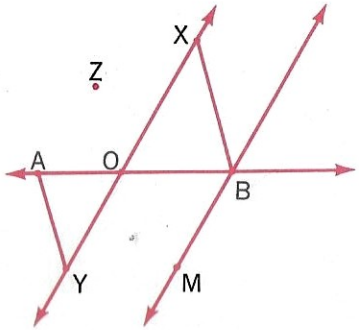
a. A-side of $\overleftrightarrow{MX}$

b. A-side of $\overleftrightarrow{MX} \cap$ X-side of $\overleftrightarrow{AM}$

c. (A-side of $\overleftrightarrow{MX} \cap$ X-side of $\overleftrightarrow{AM}$) $\cap$ M-side of $\overleftrightarrow{AX}$

d. {Points from **c**} $\cup$ $\overline{AM} \cup \overline{AX} \cup \overline{MX}$

9. In the drawing at the right, $\overleftrightarrow{XY}$ is parallel to $\overleftrightarrow{MB}$. Complete sentences **a–h**.



a. $\overrightarrow{OA} \cup \overrightarrow{OB} =$?

b. $\overrightarrow{OA} \cap \overrightarrow{OB} =$?

c. $(\overline{AO} \cup \overline{OY}) \cup \overline{AY} =$?

d. $\overleftrightarrow{XY} \cap \overleftrightarrow{MB} =$?

e. $\overline{OX} \cap \overline{BX} =$?

f. $\overline{OX} \cap \overline{AY} =$?

g. (Y-side of $\overleftrightarrow{AB}$) $\cap$ (A-side of $\overleftrightarrow{XY}$) $\cap$ (O-side of $\overleftrightarrow{AY}$) = ?

h. Point ? is on the A-side of $\overleftrightarrow{XY}$ and the X-side of $\overleftrightarrow{AB}$.

10. Use the drawing for Exercise 9. Replace the ● with $\subseteq$ (is contained in) or $\nsubseteq$ (is not contained in) to make true sentences.

a. $\overline{XB}$ ● O-side of $\overline{AY}$

b. $\overleftrightarrow{MB}$ ● B-side of $\overleftrightarrow{XY}$

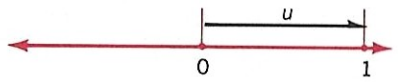
c. $\triangle AOY$ ● A-side of $\overleftrightarrow{XY}$

d. Interior of $\triangle AOY$ ● A-side of $\overline{XY}$

e. X ● A-side of $\overleftrightarrow{XY}$

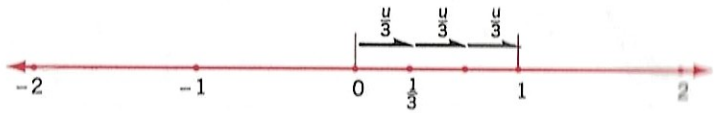
f. Z ● (A-side of $\overleftrightarrow{XY} \cap$ X-side of $\overleftrightarrow{AB}$)

4–2 Associating Points with Numbers



You know that there is a one-to-one correspondence between points on a line and the real numbers. This can be represented in many ways. Begin by choosing *any* point to match 0. Then choose a unit vector, $\vec{u}$, place its initial point at zero, and match its terminal point with a point on the line, as shown above. Call this point 1. How do you locate the point for 2? for 10? for 100? −1? −7?

To locate the point that matches $\frac{1}{3}$, separate the unit vector into three vectors of equal length. The point at the end of the first vector matches $\frac{1}{3}$.

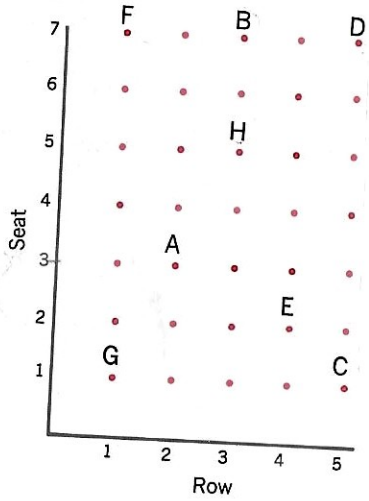


The point at the end of the second vector matches $\frac{2}{3}$; the point at the end of the third vector matches $\frac{3}{3}$, or 1. How do you locate the point that matches $7\frac{1}{3}$? $-1\frac{3}{4}$? $2\frac{5}{7}$? In the same way, you can match all

...all points on the line. But some points, infinitely many, are left over. The remaining points are matched with other real numbers, the irrational numbers, such as $\sqrt{2}$, π, $\sqrt[3]{3}$, and $-\sqrt{7}$.

Is there a relationship between the real numbers and points in a plane that is similar to the relationship between the real numbers and points on a line? In the following example, numbers are associated with *finite* sets of points not on a line.

Think of the diagram below as a classroom. The dots represent seats lined up in 5 rows, with 7 seats in a row. If the teacher asked the students in row number 3 to stand, seven people would stand up. By asking all students sitting in seat number 2, five people would stand. But by specifying *row 3, seat 2*, only one person would stand. One **ordered pair** of numbers is needed to locate each person.



Which labeled dot is named by

a. row 2, seat 3? **b.** row 5, seat 1? **c.** row 1, seat 7?

Give the row number and seat number for

d. *B*. **e.** *D*. **f.** *E*. **g.** *G*. **h.** *H*.

The pair of numbers (3, 5) is ordered: 3 is the number for the row and is first, and 5 is the number for the seat and is second. Which dot matches (3, 5)? (5, 1)? (1, 1)? (2, 3)?

The example of the classroom suggests a way to establish a one-to-one correspondence between points of the plane and ordered pairs of numbers.

Begin with a plane. Choose any line in the plane, such as $\overleftrightarrow{XX'}$, and associate real numbers with points on the line, as in Figure 1.

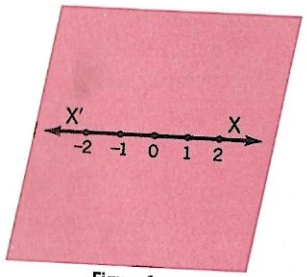


Figure 1

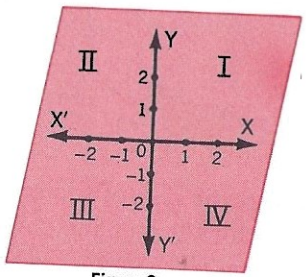
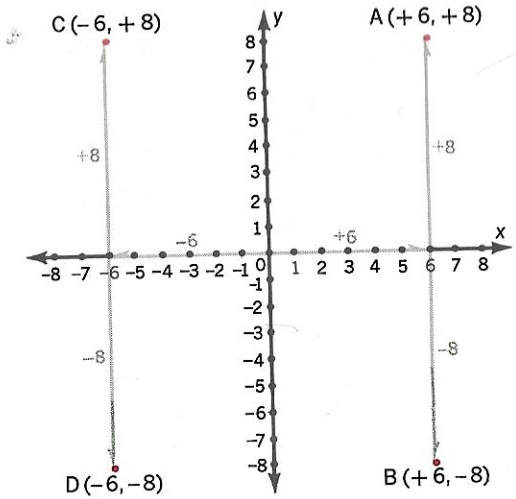


Figure 2

Now think of a line, $\overleftrightarrow{YY'}$, perpendicular to $\overleftrightarrow{XX'}$ at 0. Use the same unit as on $\overleftrightarrow{XX'}$ and associate numbers with each point on this line, as in Figure 2 at the bottom of page 168.

Now you can associate a pair of real numbers with any point in the plane by referring to $\overleftrightarrow{XX'}$ and $\overleftrightarrow{YY'}$. These lines are called **axes:** $\overleftrightarrow{XX'}$ is the horizontal axis, or x axis, and $\overleftrightarrow{YY'}$ is the vertical axis, or y axis. The point of intersection is the **origin.** The axes separate the plane into *quadrants*, as you saw in Exercise 6 on page 166.

The diagram below shows points for four ordered pairs of numbers. The point A for the pair (+6, +8) is located by vectors. Think of a +6 horizontal vector beginning at 0, and a +8 vector perpendicular to it at its terminal point. Then A is 6 units *to the right* and 8 units *up*.



B, (+6, −8): +6 vector from 0 and −8 vector perpendicular to it (6 units *to the right* and 8 units *down*).

C, (−6, +8): −6 vector from 0 and +8 vector perpendicular to it (6 units *to the left* and 8 units *up*).

D, (−6, −8): −6 vector from 0 and −8 vector perpendicular to it (6 units *to the left* and 8 units *down*).

In the ordered pair that names a point in a plane, the x number, the **abscissa,** is always first and the y number, the **ordinate,** second. The two numbers are the **coordinates** of the point. They may be called *rectangular coordinates* because of the way the point is located. The point is the graph of the number pair. A plane with number pairs associated with the points is called a **coordinate plane.**

Checkpoint

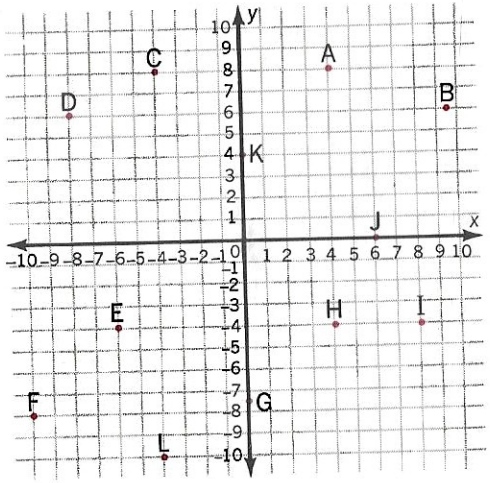
1. What is a rectangular-coordinate plane? How is a point located in this system?

2. What is the abscissa of an ordered pair? the ordinate?

Exercises

A **1.** One coordinate of each point shown on the diagram is given. What is the other coordinate?

$A(4, ?)$	$B(?, 6)$
$C(-4, ?)$	$D(?, 6)$
$E(-6, ?)$	$F(?, -8)$
$G(?, -7.5)$	$H(4, ?)$
$I(?, -4)$	$J(?, 0)$
$K(?, 4)$	$L(?, -10)$



2. Name the abscissa and ordinate for each point graphed in Figure 1 below.

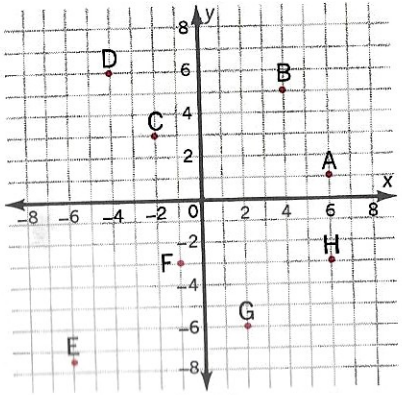


Figure 1

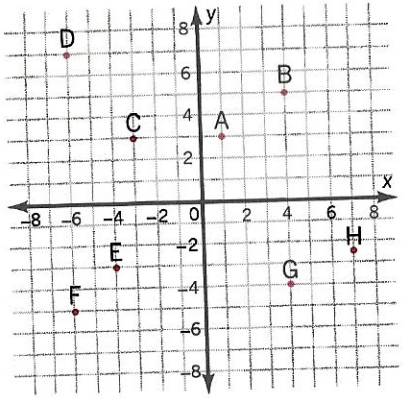


Figure 2

3. Write the coordinates for each point graphed in Figure 2 above.

4. Make your own coordinate plane on graph paper, and graph the following points.

$A(-2\frac{1}{4}, 6)$	$B(-1, -4)$	$C(0, 5)$	$D(-3, 0)$
$E(6, 6)$	$F(3, 7.5)$	$G(-3, -7)$	$H(5, -5)$
$I(-5, 5)$	$J(0, 0)$	$K(8, -2)$	$L(-5, -3\frac{1}{2})$

5. Draw axes on graph paper and graph these points: $A(3, 1)$, $B(-1, 1)$, $C(-2, -2)$, $D(2, -2)$. Connect A to B, B to C, C to D, and D to A by line segments. What is the name of the geometric figure you obtain?

6. Draw axes and graph these points: $(-6, 3)$, $(-4, 3)$, $(-1\frac{1}{2}, 3)$, $(0, 3)$, $(2.6, 3)$, $(4, 3)$. What do you observe about the ordinates of all six points? If you graphed *all* points with the ordinate of 3, your graph would be a ? parallel to the ? axis.

7. Use the same axes as in Exercise 6, and locate these points: $(-6, -4)$, $(-4\frac{3}{4}, -4)$, $(-1, -4)$, $(0, -4)$, $(3\frac{1}{3}, -4)$, $(5, -4)$. What is true about the ordinates of these six points? If you graphed *all* points with the ordinate -4, your graph would be a ? parallel to the ? axis.

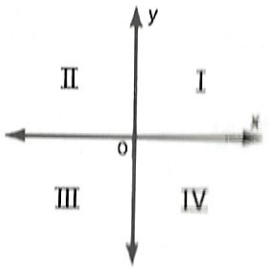
8. From your answers to Exercises 6 and 7, what statement can you make about the graph of all points with the same ordinate?

9. Draw axes on graph paper and graph these points: $(4, 5)$, $(4, 3)$, $(4, \frac{1}{2})$, $(4, 0)$, $(4, -1)$, $(4, -4\frac{2}{3})$, $(4, -6)$. What pattern do you see in the coordinates for these six points? If you graph *all* the points with 4 as their abscissa, the graph will be a ? parallel to the ? axis.

10. Using the same axes as in Exercise 9, graph these points: $(-2, -5.75)$, $(-2, -4)$, $(-2, -1)$, $(-2, 0)$, $(-2, 2)$, $(-2, 3\frac{1}{2})$, $(-2, 6)$. What is alike for the coordinates of these points? What will be the graph of *all* points with the abscissa -2?

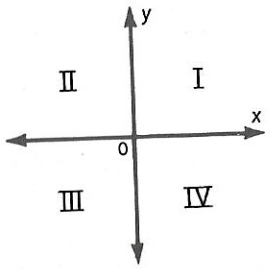
11. From your answers to Exercises 9 and 10, what statement can you make about the graph of all points with the same abscissa?

12. The figure at the right shows the four quadrants into which the coordinate plane is separated. Both the abscissa and ordinate of any point in the first quadrant are positive. Describe the abscissa and ordinate of any point in each of the other three quadrants.



B **13.** Referring to the figure below, complete the following sentences with I, II, III, or IV.

a. $\{x : x > 0 \cap y > 0\} = \underline{?}$
b. $\{x : x > 0 \cap y < 0\} = \underline{?}$
c. $\{x : x < 0 \cap y > 0\} = \underline{?}$
d. $\{x : x < 0 \cap y < 0\} = \underline{?}$



14. If the x axis in the figure for Exercise 13 is $\overleftrightarrow{XX'}$ and the y axis is $\overleftrightarrow{YY'}$, complete the following to make true sentences.

a. $\overleftrightarrow{YY'} \cap \overleftrightarrow{XX'} = \underline{?}$
b. $\overleftrightarrow{YY'} \cap \text{I} = \underline{?}$
c. $\text{I} \cap \text{II} = \underline{?}$
d. $\text{I} \cup \text{II} \cup \text{III} \cup \text{IV} \cup \overleftrightarrow{XX'} \cup \overleftrightarrow{YY'} = \underline{?}$

15. The following groups of coordinates each represent three vertices of a rectangle. Graph the points in each group. Then locate the fourth vertex and give its coordinates.

a. $(-3, -3), (-7, -5), (-7, -3)$
b. $(-7, 3), (-4, 3), (-7, 5)$
c. $(6, 8), (7, 2), (7, 8)$
d. $(0, 3\frac{1}{2}), (6, -4), (6, 3\frac{1}{2})$
e. $(-6, 1), (-1, 5), (-6, 5)$
f. $(-5, -1), (-8, -4), (-6, -6)$

16. The following groups of coordinates each represent three vertices of a parallelogram. Graph the points in each group. Then locate two points, each of which could be the fourth vertex. Give the coordinates of both points.

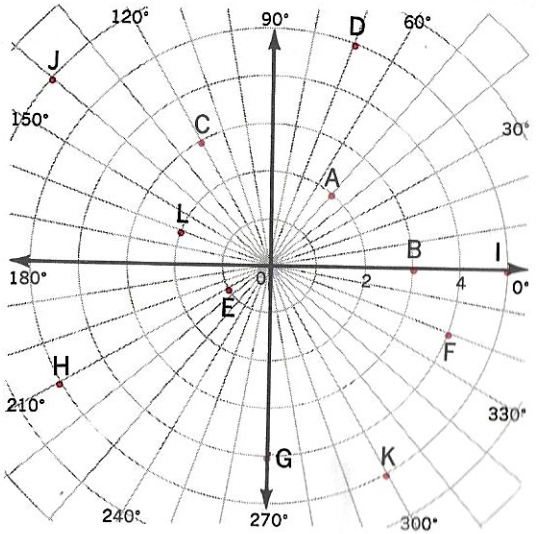
a. $(-3, -3), (-7, -5), (-7, -3)$
b. $(-7, 3), (-4, 3), (-7, 5)$
c. $(6, 8), (7, 2), (7, 8)$
d. $(3, 4), (2, 7), (3, 7)$
e. $(-6, 1), (-1, 5), (-6, 5)$
f. $(-6, 8), (-8, 5), (-1, 6)$

17. The following pairs of coordinates locate the endpoints of the hypotenuse of a right triangle. There is only one point below the hypotenuse that could be the third vertex if the triangle is to be a right triangle. Give the coordinates of that point.

a. (0, 6) and (5, 0)
b. $(-5, 0)$ and $(0, -3)$
c. (4, 4) and $(1, -2)$
d. $(-2, 7)$ and $(2, -2)$

C **18.** Points in a plane can also be located by a distance on a ray and an angle measure in degrees.

Designate a horizontal ray pointing to the right as the **zero ray.** This ray can be rotated in a circle, sweeping over 360°. Thus, $\frac{1}{360}$ of a rotation is equivalent to 1°. Assign a ray that has made $\frac{1}{36}$ of a rotation as the 10° ray; a ray that has made $\frac{1}{18}$ of a rotation as the 20° ray, and so on. The zero ray is assigned to 0°. Assign units to the zero ray as you would a number line. The endpoint of the zero ray, the **pole,** is 0. Then with the pole as the center, draw a circle with a radius of 1 unit, a circle with a radius of 2 units, and so on. This model represents the **polar-coordinate plane.**



To locate a point, move along the zero ray, or **polar axis,** a certain number of units and then move along in a circle to the ray for the correct degree measure. As with the rectangular-coordinate plane, points are named by ordered pairs. The point for (2, 50°) is point A in the diagram. What is the name for point B? It is at the point for 3 on the zero ray, so it would be (3, 0°).

Give the ordered pairs for points C–L.

4–3 Graphing Relations

A space vehicle is moving through space at 7 miles per second. The total distance it travels can be found by multiplying by 7 the number of seconds it is moving.

This is an example of a **relation.** In mathematics, a relation, such as that between the d miles that a space vehicle travels in t seconds at 7 miles per second, can be expressed in several ways, any one of which is correct.

① A sentence: The distance traveled in miles is 7 times the number of seconds.

② An equation: $d = 7t$.

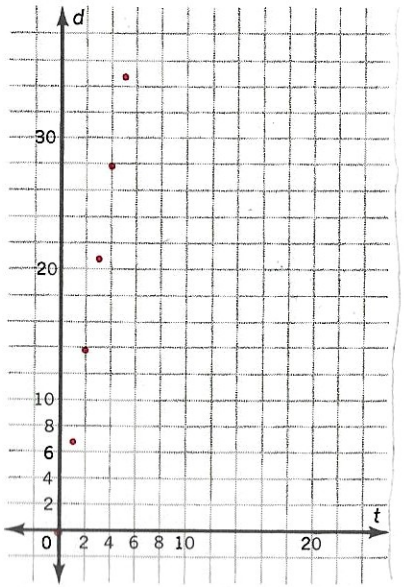
③ Ordered pairs of numbers in a table.

t	0	1	2	3	4	5	6 ···
d	0	7	14	21	28	35	42 ···

④ Ordered pairs of numbers in set form.

$$\{(0, 0), (1, 7), (2, 14), (3, 21), (4, 28), (5, 35), (6, 42), \cdots\}$$

⑤ Ordered pairs of numbers, (t, d), graphed as points in a plane.



Here is another relation. A railroad divided its line into zones and charged one fare to any station in a zone. For \$0.50 you could ride to station 1, 2, or 3. For \$0.75 you could ride to station 4 or 5. For \$1.10 you could ride to station 6, 7, or 8.

This relation cannot be expressed using an equation. You could use the graph, but graphs are often not compact. In this book, a relation is a set of ordered pairs as in the table or in set notation.

$$\{(0, 0), (1, 7), (2, 14), (3, 21), (4, 28), (5, 35), (6, 42), \cdots\}$$

Definition A relation is a set of ordered pairs.

The **domain** of a relation is the set of first numbers of the ordered pairs. The **range** is the set of second numbers in the pairs.

In the space-vehicle example, the domain is $\{0, 1, 2, 3, 4, \cdots\}$ and the range is $\{0, 7, 14, 21, 28, \cdots\}$. The problem is such that the domain could also be defined as the set of nonnegative real numbers.

In the train-fare example, if the domain is $\{\$0.50, \$0.75, \$1.10\}$, what is the range?

The graph of a relation is the set of points whose coordinates are the ordered pairs of the relation.

To graph a relation defined by an equation, plot the points whose coordinates are the pairs of numbers that make the equation true.

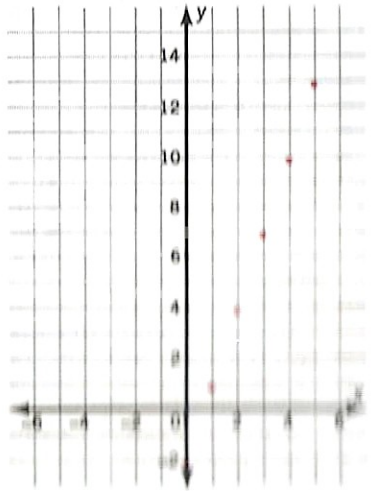
EXAMPLE. Graph the relation defined by $y = 3x - 2$ if the replacement set for x, or domain, is $\{0, 1, 2, 3, 4, 5\}$. Give the range.

① Determine the ordered pairs of the relation.

When x is	0	1	2	3	4	5
y is	−2	1	4	7	10	13

Notice that the range is $\{-2, 1, 4, 7, 10, 13\}$.

② Set up coordinate axes and locate a point for each pair. The six points that form the graph are shown at the right.



Since the domain in the above example was a finite set of whole numbers, the graph consisted of a finite set of points. If the domain were given as the set of nonnegative real numbers, as it was for $d = 7t$, then there would be infinitely many points and the graph would be a line, as will be shown in a later section.

Checkpoint

1. What is a relation? Give some examples of how it may be defined.

2. A relation has a domain and a range. Explain these terms.

Exercises

A — Give the domain and range of each relation defined below. Draw the graph of each.

1. $\{(1, 2), (2, 2), (3, 1), (4, 3), (5, 3), (6, 6)\}$

2. $\{(1, 1), (3, 3), (6, 6)\}$

3. $\{(1, 1), (3, 3), (4, 4), (5, 3), (6, 1), (6, 7), (7, 5)\}$

4. The equation $y = 3x + 4$ defines a relation between x and y. Complete the table showing the number pairs that satisfy the relation $y = 3x + 4$, if the domain is $\{-1, 0, 1, 2, 3, 4, 7\}$.

When x is	-1	0	1	2	3	4	7
y is	1	4	?	?	?	?	?

5. The perimeter of a regular hexagon is related to the length of one side. The relation is defined by $p = 6s$. Complete the table below for $p = 6s$. Then graph the relation.

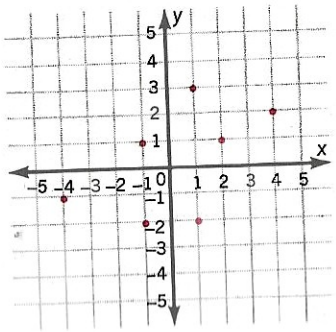
When s is	0	1	2	3	4	5	6
p is	0	6	?	?	?	?	?

6. The ordered pairs listed below define a relation between the two sets $\{0, 2, 3, 4, 5\}$ and $\{1, 2, 3, 4\}$.

$\{(0, 1), (2, 1), (3, 2), (4, 2), (4, 3), (5, 4)\}$.

Graph the relation.

7. The graph at the right defines a relation. Define this relation by listing the ordered pairs that are the coordinates of the points.



8. The relation in Exercise 7 cannot be defined easily by a simple sentence, but it can be defined easily by a table. Construct such a table for this relation.

9. Draw the graph of the relation defined by $y = 2x + 1$ where x is $\{-1, -2, -3, -4, -5\}$.

— Make a table showing the ordered pairs of each relation. Then graph the relation. The domain given for the variable is to be represented on the horizontal axis.

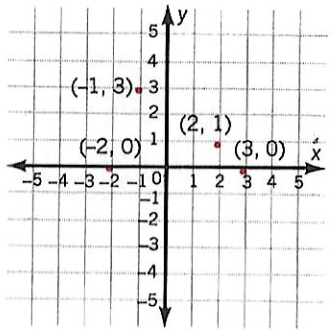
Equation	Domain
10. $\{(x, y) : y = 2x\}$	$x \in \{0, 2, 4, 6\}$
11. $\{(x, y) : y = 2x + 3\}$	$x \in \{0, 2, 4, 6\}$
12. $\{(x, y) : y = x^2\}$	$x \in \{-3, -2, -1, 0, 1, 2, 3\}$
13. $\{(x, y) : y = -x^2\}$	$x \in \{-3, -2, -1, 0, 1, 2, 3\}$
14. $\{(x, y) : xy = 24\}$	$x \in \{1, 2, 3, 4, 6, 8, 12, 24\}$
15. $\{(x, y) : x + y = 10\}$	$x \in \{-5, -3, -1, 1, 3, 5\}$

16. Which relations in Exercises 10–15 have graphs that lie along a line? What do the equations for these relations have in common?

B **17.** A relation defined by the equation $y = |2x + 5|$ has the domain $\{-1, -2, -3, -4, -5\}$. What is its range?

— Give the domain and range of each of the graphed relations.

18.



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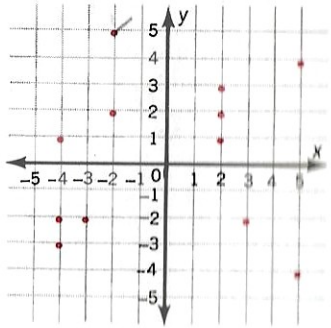
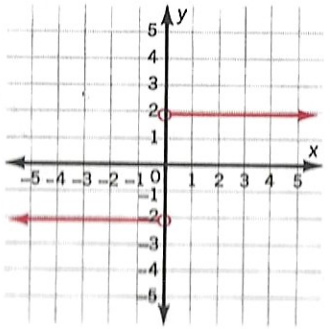


C — Give the domain and range of each of the graphed relations.

20.



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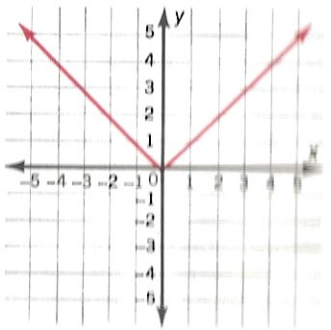


4–4 Graphing Linear Functions

A relation is a set of ordered pairs.

a. {($0.50, 1), ($0.50, 2), ($0.50, 3), ($0.75, 4), ($0.75, 5), ($1.10, 6), ($1.10, 7), ($1.10, 8)}

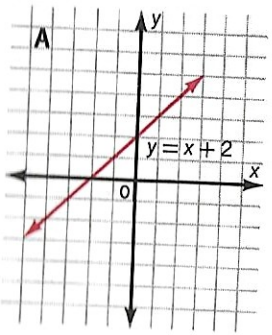
b. {(0, 0), (1, 7), (2, 14), (3, 21), (4, 28), (5, 35), (6, 42)}

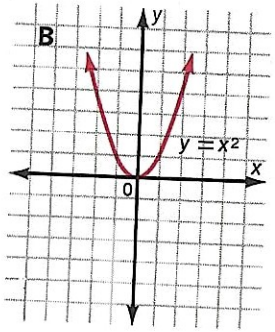
The first elements of the ordered pairs of some relations, such as **b**, are always different. Such relations are called **functions.**

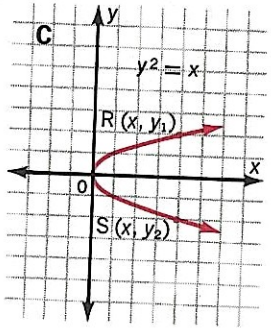
Definition A function is a set of ordered pairs of which no two first elements are the same.

If a function is expressed as an equation, the variables can be classified as dependent or independent. The **independent variable** is the variable whose replacement set is the domain of the function. The value of the **dependent variable** is found by choosing values from the replacement set for the independent variable and solving the equation; that is, the value of one variable *depends* on the value chosen for the other variable.

Below are the graphs of three relations.







If you examine the ordered pair for any point on the graph in A, the first member will always be different from the first member in any other ordered pair. The same is true in B. So A and B are the graphs of functions. But in C, the ordered pairs for points R and S have the same first member, that is, the same value for the independent variable, x. In fact, except for (0, 0), there is an infinite number of pairs of ordered pairs with the same first member. Explain why.

It is easy to see from a graph whether or not the relation is a function. Think of a vertical line moving from left to right. If the line in every position intersects the graph in only one point, the relation graphed is a function.

A function of special importance is the **linear function.** It is *linear* because its graph is a line. The function graphed in A is a linear function. It will be defined in algebraic terms in a later section.

EXAMPLE 1. Graph $y = x + 4$ if the domain is the real numbers.

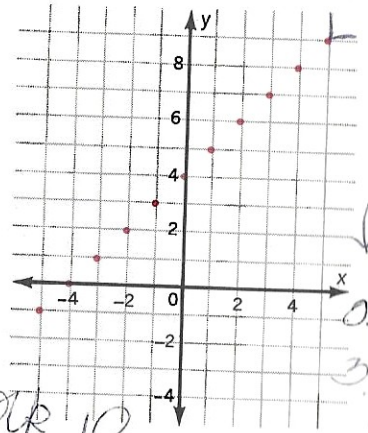
① Make a table of some pairs of numbers that satisfy $y = x + 4$. Choose any numbers for x and find the corresponding numbers for y.

When x is	−5	−4	−3	−2	−1	0	1	2	3	4	5
y is	−1	0	1	2	3	4	5	6	7	8	9

The table shows only eleven of the infinite set of ordered pairs that make $y = x + 4$ true.

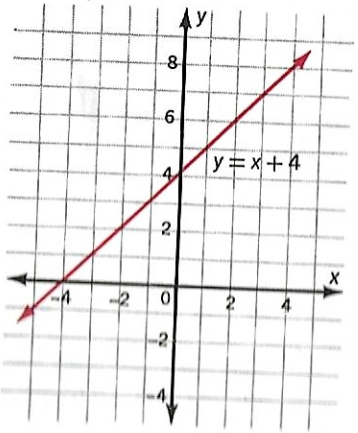
② On coordinate axes, locate a point for each ordered pair in the table.

The points are shown at the right.



③ The eleven points lie in a line. If you were to locate a point for every ordered pair that satisfies the equation, all the points would form a line. *The line is the graph of* $y = x + 4$.

Use a straightedge and draw the part of the line that includes the set of points represented in the table. The complete graph of $y = x + 4$ continues indefinitely in both directions. The arrows show that the graph is a line.

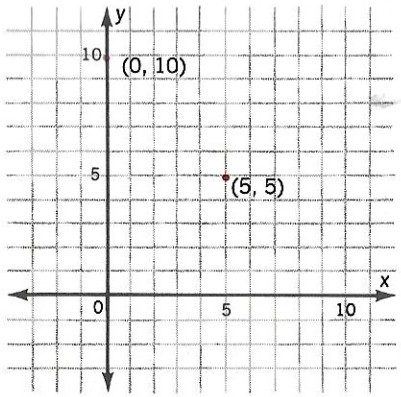


If you know that an equation defines a linear function, you actually need to plot only two points (with a third point as a check).

EXAMPLE 2. $x + y = 10$ defines a linear function if the domain for x is the set of real numbers. Graph the function $\{(x, y) : x + y = 10\}$.

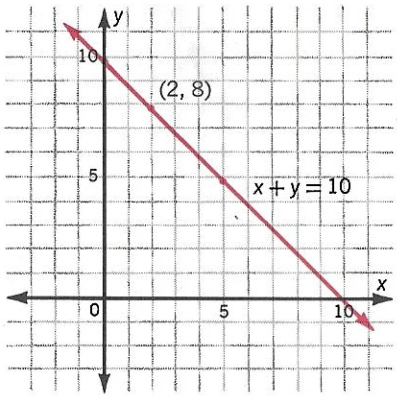
① Find two ordered pairs that satisfy the equation. (It is wise to choose values so that the corresponding points are not too close together.)

$\{(0, 10), (5, 5)\}$



② Plot, or locate, the points.

③ Draw the line containing the two points.



④ Check another pair of numbers to verify that you have the correct line for the linear function. For example, (2, 8) makes $x + y = 10$ true, and the point is on the graph.

Checkpoint

1. How is a function a special relation?
2. What is a linear function?

Exercises

A — In Exercises 1–10, tell which ordered pair satisfies the equation.

1. $y - x = 6$; $(-2, 8)$, $(2, 8)$
2. $x - 3y = 12$; $(3, 5)$, $(6, -2)$
3. $2y + 6x = 0$; $(0, 0)$, $(6, 2)$
4. $4x - 5y = 1$; $(0, 0)$, $(-1, -1)$
5. $\frac{y}{5} = 2x$; $(1, -10)$, $(-2, -20)$
6. $\frac{y}{4} = x$; $(12, -3)$, $(-1, -4)$
7. $2(x - 3) = 4(y - 2)$; $(0, -\frac{1}{2})$, $(-1, 0)$
8. $1\frac{1}{2}x + y = 2$; $(0, 2)$, $(2, 1)$
9. $5x - 3y - 30 = 0$; $(6, 0)$, $(-3, 15)$
10. $3y = 7x$; $(0, 1)$, $(0, 0)$
11. $y = 2x - 6$ defines a function. Complete the following table.

When x is	-4	-2	0	2	4	6	10
y is	-14	?	?	?	?	?	?

Draw the graph for the function if the domain is the set of real numbers, using the ordered pairs in the table.

12. Why does $y = 2x - 6$ in Exercise 11 define a linear function?
13. Does $y = x - 3$ define a linear function? Explain.

Make a table containing several pairs of numbers for the function defined in each of the following sets. The domain is the set of real numbers. Draw the graph of each function and tell if it is a linear function.

14. $\{(x, y) : y = x + 2\}$
15. $\{(x, y) : y = 3x - 2\}$
16. $\{(x, y) : y = 5x + 6\}$
17. $\{(x, y) : y = 4x\}$
18. $\{(x, y) : y = 4x - 5\}$
19. $\{(x, y) : y = \frac{1}{2}x + 3\}$
20. $\left\{(x, y) : y = \frac{6 - 2x}{3}\right\}$
21. $\left\{(x, y) : y = \frac{-2x + 1}{5}\right\}$

B — Draw the graph of each equation. The replacement set is the set of real numbers.

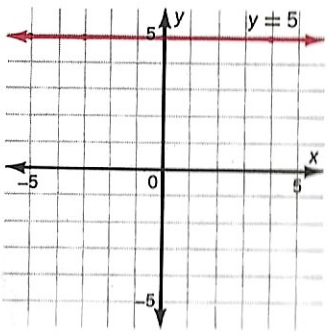
Example. Graph $\{(x, y) : y = 5\}$.

① Think $y = (0 \cdot x) + 5$.

When x is	-5	-3	0	4
y is	5	5	5	5

② Graph the points.

③ The graph is a line parallel to the x axis, crossing the y axis at (0, 5).



22. $y = (0 \cdot x) - 4$
23. $y = (0 \cdot x) + 7$
24. $x = 3$
25. $x = 6\frac{1}{2}$
26. $y = -9$
27. $x = -8$
28. Which equations in Exercises 22–27 define a relation? a function? Equations of the form $y = b$, where b is a constant number, define the **constant function.**

4–5 Equations for Linear Functions

You know from the previous section that the following equations define linear functions if x is in the set of real numbers.

a. $y = x + 4$ **b.** $y = -x + 10$ **c.** $y = 2x - 6$

From these examples, the following definition is stated.

Definition A linear function is a function defined by the equation $y = mx + b$, where m and b are constants ($m \neq 0$) and the domain for x is the set of real numbers.

What is m in sentences **a** and **b** above? What is b in sentence **a**? in sentence **c**?

To see if an equation defines a linear function, write an equivalent equation in the form $y = mx + b$.

EXAMPLE 1. Write an equivalent equation for $2x - 3y = 12$ in the form $y = mx + b$. Then graph the equation.

① For convenience, make y have a positive coefficient by multiplying both sides of the equation by -1.

$$-1(2x - 3y) = -1(12)$$
$$-2x + 3y = -12$$

② Add $2x$.

$$-2x + 3y + 2x = -12 + 2x$$
$$3y = -12 + 2x$$

③ Multiply by $\frac{1}{3}$.

$$\tfrac{1}{3}(3y) = \tfrac{1}{3}(-12 + 2x)$$
$$y = -4 + \tfrac{2}{3}x,$$
$$\text{or } y = \tfrac{2}{3}x - 4$$

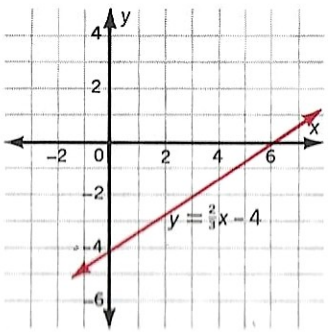
Then $2x - 3y = 12$ is equivalent to $y = \frac{2}{3}x - 4$, which is in the form $y = mx + b$, and it is a linear function.

If x is 0, then y is $0 - 4$, or -4.

The table shows some other ordered pairs.

x	0	3	−3	6	−6
y	−4	−2	−6	0	−8

Locate the points on the coordinate plane. The graph is a line, as shown at the right.



EXAMPLE 2. Express $2x + 5y = -8$ in the form $y = mx + b$ and graph the equation.

$$2x + 5y = -8$$

① Add $-2x$.

$$2x + 5y - 2x = -8 - 2x$$
$$5y = -8 - 2x$$

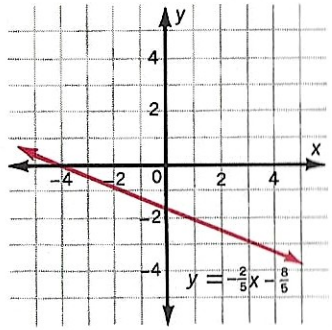
② Divide by 5.

$$y = \frac{-8 - 2x}{5}$$
$$y = -\tfrac{8}{5} - \tfrac{2}{5}x, \text{ or } -\tfrac{2}{5}x - \tfrac{8}{5}$$

③ Find some ordered pairs of the function. Try to choose integers for x such that the corresponding values for y will be integers. When x is -9, y is $-\frac{2}{5}(-9) - \frac{8}{5}$, or $\frac{18}{5} - \frac{8}{5}$, or 2. Check to see that the other ordered pairs in the table are correct.

x	−9	−4	1	6
y	2	0	−2	−4

④ Use the ordered pairs to draw the graph.



Now try these

Write the following equations in $y = mx + b$ form.

1. $2x + y = 1$ **2.** $x + 6y = 9$

3. $4x - \frac{1}{2}y = 3$ **4.** $-\frac{1}{5}x + \frac{1}{3}y = -2$

5. Which of the following are linear functions?

a. $3x(2y) = -8$ **b.** $15 - \frac{3}{4}y = 3x$

c. $-x + 4 = 7y$ **d.** $y = 1.6 - 2x^2$

Answers: **1.** $y = -2x + 1$ **2.** $y = -\frac{1}{6}x + 1\frac{1}{2}$ **3.** $y = 8x - 6$ **4.** $y = \frac{3}{5}x - 6$ **5.** **b** and **c**

Checkpoint

1. Define *linear function.*

2. How do you know if an equation defines a linear function?

Exercises

A — Write each equation in $y = mx + b$ form.

1. $2x + y = 3$
2. $3x + y = 4$
3. $2x + 3y = 7$
4. $x + 2y = 5$
5. $2x - y = 3$
6. $3x - y = 4$
7. $5x - y = -6$
8. $3x - 2y = 4$
9. $2x - 3y = -7$
10. $5x - 7y = 12$
11. $3x + 4y = -5$
12. $4x - 3y = 5$

— Find three ordered pairs that satisfy each of the following linear functions. The replacement set is the set of real numbers. Graph each function.

13. $2x + y = 3$
14. $3x + y = 4$
15. $2x + 3y = 7$
16. $3x + 2y = 5$
17. $2x - y = 3$
18. $3x - 2y = 0$
19. $2x - 3y = -7$
20. $5x - 7y = 12$
21. $3x + 4y = -5$
22. $4x - 3y = 5$

— Draw the graphs for the following sets of number pairs. The replacement set is the set of real numbers. Represent the variable that is first in each pair on the horizontal axis.

23. $\{(x, y) : x + y = 5\}$
24. $\{(x, y) : 3x + 2y = -1\}$
25. $\{(x, y) : \frac{x}{2} + y = 3\}$
26. $\{(x, y) : 3(x + 2) = 2(y + 1)\}$
27. $\{(a, b) : a = b + 1\}$
28. $\{(s, t) : -2s + t = 3\}$

— Find the coordinates of the point at which the graph of each equation crosses the x axis (y will be 0) and the coordinates of the point at which the graph crosses the y axis (x will be 0).

29. $y = 2x - 6$
30. $9y - 26 = 2x$
31. $2x + 3y = 6$
32. $3x + y = 4$
33. $y = -2x$
34. $3(2x - 4) = 2(y - 5)$
35. $x - y = 10$
36. $3\left(\frac{x}{2}\right) + y = 3$
37. $12x - 5 = 5y$
38. $\frac{x}{6} = 3x + 4$

B

39. What is the graph of the equation $x = 0$? of $x = 1$? $x = -2$? If b is any constant, does $x = b$ define a linear function?

40. What is the graph of the equation $y = 0$? What is the graph of $y = -3\frac{1}{2}$? $y = 5$? If b is any constant, does $y = b$ define a function? Does $y = b$ define a linear function?

41. Complete this table showing seven ordered pairs that satisfy $y = x^2 + 2$. Then graph the ordered pairs.

x	−5	−3	−1	0	1	3	5
y	27	11	?	?	?	?	?

Does $y = x^2 + 2$ define a linear function? Sketch a smooth curve containing the seven points you plotted.

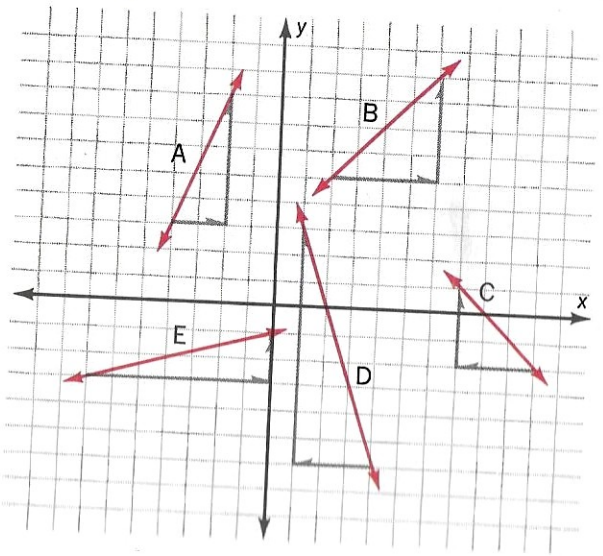
42. Complete the table showing six number pairs that satisfy $xy = 6$. Then plot the six points.

x	12	6	3	2	1	$\frac{1}{2}$
y	$\frac{1}{2}$	1	?	?	?	?

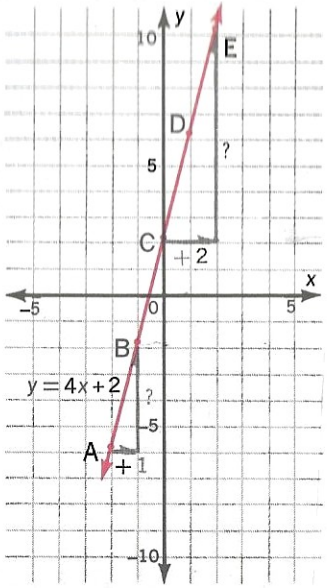
Do the points lie in a line? Does $xy = 6$ define a function? a linear function? Sketch a smooth curve containing the six points.

4–6 Slope of a Line

If you choose any two points on a line, the segment between the two points slants in a particular direction. Segments of the same line all slant in the same direction. To get from one point on the line to another, you could move along the line. You could also move between the two points, first in a horizontal direction and then in a vertical direction, as shown by the graphs below.



Consider the equation $y = 4x + 2$. The graph of the equation is shown at the right below.



Start at point A and draw a $+1$ vector horizontally. What vertical vector then brings you to B? Start at B and draw a $+1$ vector horizontally. What vector then brings you to C? If you start at any point and draw a $+1$ vector horizontally, what vertical vector brings you to a point of the line?

All along the line, as x changes by $+1$, y changes by ?. The ratio of the number for which the vertical vector is a model, $+4$, to the number for which the horizontal vector is a model, $+1$, is ?. This can be abbreviated as

$$\frac{\text{vertical vector}}{\text{horizontal vector}} = \frac{+4}{+1}.$$

If you start at point C and draw a $+2$ vector horizontally, what vertical vector brings you to a point on the line? If you start at any point on the line and draw a $+2$ vector horizontally, what vertical vector brings you to a point of the line?

As x changes by $+2$, y changes by ?. The ratio of the number for which the vertical vector is a model, $+8$, to the number for which the horizontal vector is a model, $+2$, is ?. This could also be expressed as the ratio of the vertical change to the horizontal change, or the ratio of the y change to the x change.

You can see that the line rises 4 times as fast as it moves to the right (4 units up for 1 unit to the right; 8 units up for 2 units to the right). Then, the **slope** of this line is $+4$. The slope is a number that indicates the slant, or steepness, of a line.

know ↑

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x}$$

Examine the following table of number pairs that satisfy the equation $y = 4x + 2$.

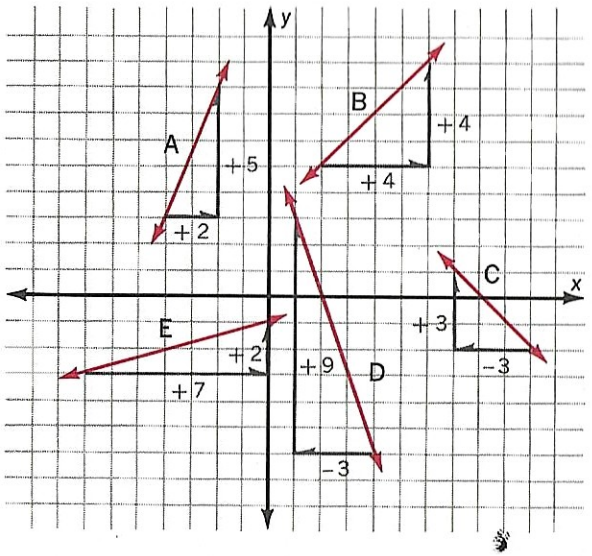
change in x →		$-1-(-2)=1$		$2-(-1)=3$	$+2$	$+1$	$+1$	$+2$	$+3$
x	-2		-1		2	4	5	6	8	11
y	-6		-2		10	18	22	26	34	46
change in y →		$-2-(-6)=4$		$10-(-2)=12$	$+8$	$+4$	$+4$	$+8$	$+12$	

From the table it can be seen that the ratio of the change in y to the change in x is always the same, $+4$.

$$\frac{\text{change in } y}{\text{change in } x} = \frac{+4}{+1}, \text{ or } \frac{+8}{+2}, \text{ or } \frac{+12}{+3}$$

If you are given the coordinates of two points, (x_1, y_1) and (x_2, y_2), you can find the change by subtracting the coordinates, rather than by graphing. The slope of a line is $\frac{y_2 - y_1}{x_2 - x_1}$, as shown in the table.

Shown below are the graphs of the lines that you saw at the beginning of the section. The equation for A is $y = 2\frac{1}{2}x + 13$; the slope is $\frac{+5}{+2}$, or $2\frac{1}{2}$. The equation for C is $y = -x + 8$; the slope is $\frac{+3}{-3}$, or -1. The equation for B is $y = x + 3$; for D, $y = -3x + 6$; for E, $y = \frac{2}{7}x - 1$. What are the slopes for B, D, and E?



Do you see a pattern between the numbers for the slopes and the equations of the lines? The slope of the line is the coefficient of x for a linear equation with y alone on one side. In other words, **the slope of $y = mx + b$ is m.**

If you compare the slopes and the lines, you will see that the lines with positive slopes slant up to the right. How would you describe the lines with negative slopes?

Another characteristic of a graph can be read from its equation. The point at which a line crosses the y axis is called the y **intercept.**

If you extended each of lines A–E on the previous page so they crossed the y axis, you could give the coordinates of the y intercept. You can do this by simply laying a straightedge along the line and seeing where it crosses the y axis.

	Equation	Slope	y intercept
A	$y = \frac{5}{2}x + 13$	$\frac{5}{2}$	(0, 13)
B	$y = x + 3$	1	?
C	$y = -x + 8$	-1	(0, 8)
D	$y = -\frac{3}{2}x + 6$	$-\frac{3}{2}$	?
E	$y = \frac{2}{7}x - 1$	$\frac{2}{7}$	?

What are the y intercepts for B, D, and E?

The y coordinate of the ordered pair for the y intercept is what constant in the equation for the line?

In general **the y intercept for $y = mx + b$ is $(0, b)$.** To understand this further, use 0 for x in the general equation.

$$y = (m \cdot 0) + b$$
$$y = b$$

Since the x coordinate is always zero at the point where a line crosses the y axis, you can see algebraically that the y coordinate is always b.

Now try these

— Complete the table.

Equation	Slope	y intercept
1. $y = 2x + 5$	?	?
2. $y = 6x - 7$	?	?
3. $y = \frac{2}{3}x + \frac{1}{2}$	?	?
4. $5x + y = 6$	?	?

Answers: **1.** 2, (0, 5) **2.** 6, (0, −7) **3.** $\frac{2}{3}$, $(0, \frac{1}{2})$ **4.** −5, (0, 6)

Checkpoint

1. What is meant by the *slope of a line*? When is the slope positive? negative?

2. How could you tell the slope of a line if you were given its graph? its equation?

3. What is the y intercept of a line?

Exercises

A — What is the slope of the graph for each equation below?

1. $y = 2x$	**2.** $y = \frac{3}{4}x + 7$	**3.** $y = -2x + 9$
4. $y = -\frac{1}{3}x$	**5.** $y = 5x$	**6.** $y = -\frac{3}{5}x - 2$
7. $y = 3x - 4$	**8.** $y = -1\frac{1}{4}x - \frac{1}{2}$	**9.** $y = x - 5$
10. $y = 4x + 1$	**11.** $y = -9x$	**12.** $y = \frac{2}{3}x - 3$

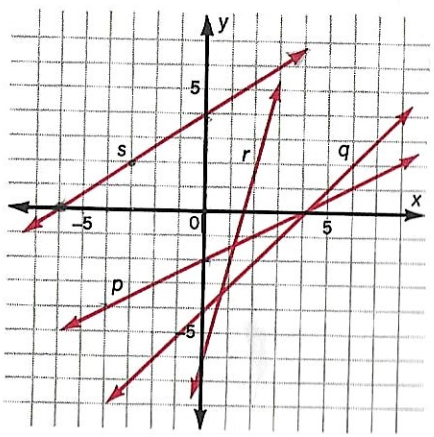
— Change each equation to $y = mx + b$ form, and then give the slope and the y intercept of the graph for the equation.

13. $2x - 5y = 8$	**14.** $y - 5x = 10$	**15.** $3y - x = 7$
16. $y - \frac{1}{2}x = 8$	**17.** $4x - 2y = 12$	**18.** $8 + y = 6x$
19. $y - 2x = -4$	**20.** $8 + 4x = 3y$	**21.** $3x - 4y = -7$
22. $5 - 2y = 5x$	**23.** $y - x = 0$	**24.** $x - y = 0$

25. Graph the points with the coordinates (−3, −14), (−1, −8), (1, −2), (3, 4), and (5, 10). Draw a line through the five points. As x changes by +2, y changes by ?. What is the slope of the line? The equation of the line is $y = ?\ x - 5$.

26. What is the equation of a line that has a slope of 3 and that crosses the y axis at (0, 5)?

27. What is the slope of each of the lines graphed below?



28. Refer to the figure for Exercise 27 to complete the following.

a. The equation for line p is $y = ?\ x - 2$.

b. The equation for line q is $y = ?\ x - 4$.

c. The equation for line r is $y = ?\ x - 6$.

d. The equation for line s is $y = ?\ x - ?$.

B — In each of the following exercises, you are given the equation for a function and its domain. Draw the graph of each function. Then describe the graph and state the range for each function.

29. $y = 4x$; $\{0, 1, 2, 3, 4\}$

30. $y = x - 8$; $\{4, 7, 9\}$

31. $y = 2x + 4$; {real numbers 2 through 8}

32. $y = -6$; {real numbers -4 through $+4$}

33. At a supermarket, chewing gum sells at 3 packages for 10¢ or 5¢ each if not bought in threes.

a. Complete the table below showing the cost in cents of x packages of gum if x is 10 or less.

When x is	0	1	2	3	4	5	6	7	8	9	10
Cost is	0	5	10	10	?	?	?	?	?	?	?

b. Plot the eleven points taken from the completed table.

c. What is the domain?

d. What is the range?

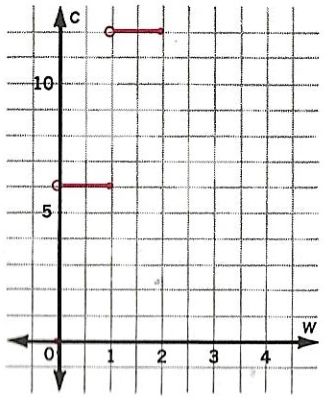
34. The cost of first-class mail is 6¢ per ounce or fraction of an ounce. Thus, the cost of mailing a letter weighing $2\frac{1}{4}$ ounces is 3×6¢, or 18¢.

a. Complete the table that shows the cost in cents of mailing a letter weighing w ounces.

When w is	0	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	$1\frac{7}{8}$	2	$2\frac{1}{8}$	3	$3\frac{1}{2}$	4	5
Cost is	0	6	6	6	?	?	?	?	?	?	?	?

b. If your letter weighs 5 ounces or less, then the cost will be any one of the numbers ?.

c. Part of the graph is shown at the right. Complete the graph for w that are equal to 0 through 4.



35. Canned cherries sell at 3 cans for 49¢ or 17¢ for each can. Make a table that shows the cost of n cans if n is 10 or less. Graph the points.

4–7 Graphing by Slopes

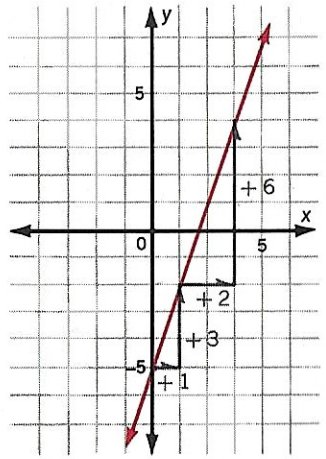
You have graphed linear functions by choosing several ordered pairs of numbers. You can also draw the graph if you know only the slope of the line and the coordinates for one point.

EXAMPLE 1. Draw the graph of a line that has a slope of 3 and passes through the point $(0, -5)$.

Locate $(0, -5)$. Since the slope is 3, or $\frac{3}{1}$, y changes by 3 as x changes by 1. From $(0, -5)$, move 1 unit to the right and then 3 units up. This locates a second point and you can draw the line through two points.

Could you also have correctly located a second point by moving 2 units to the right and then 6 units up? Why?

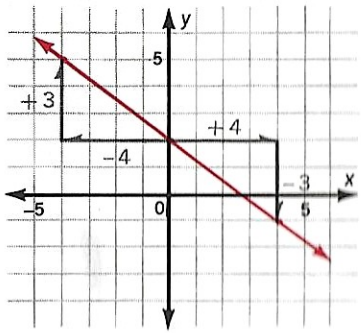
Locate one or two more points to check your work.



EXAMPLE 2. Use a point and the slope to graph $y = -\frac{3}{4}x + 2$.

Find the coordinates of one point on the line. The simplest one is found when x is 0. The point is $(0, 2)$. Locate that point.

From the equation, the slope is $-\frac{3}{4}$. You can interpret the $-\frac{3}{4}$ as meaning y changes by -3 as x changes by $+4$, $\frac{-3}{+4}$, or y changes by $+3$ as x changes by -4, $\frac{+3}{-4}$. Locate a point and draw the graph. You can see from the graph that both points are on the line and either one would give the correct line.



Notice that one of the changes is negative and the other positive. The minus sign in front of the fraction for the slope could not mean that both changes were negative, because that would give a positive slope; that is, $\frac{-3}{-4}$ is $+\frac{3}{4}$, which is not the slope of this line.

Exercises

A — In each of the following exercises, you are given the coordinates of a point and the slope of a line. Use them to draw a line.

1. $(0, 5)$; 4
2. $(0, 0)$; $-\frac{1}{2}$
3. $(-3, -2)$; 3
4. $(0, -3)$; -6
5. $(3, 4)$; $\frac{2}{3}$
6. $(-5, 7)$; $-\frac{3}{5}$

— Draw the graph for each of the following equations by using a point and the slope. Be sure each equation is in the $y = mx + b$ form.

7. $y = 3x + 2$
8. $y = -6x$
9. $y = 5x - 6$
10. $-2x + y = 4$
11. $y = -\frac{2}{3}x - 6$
12. $3x = y + 6$
13. $y - 3x = 1$
14. $4y = 5x + 12$
15. $2x - y = 3$
16. $-5y + 2x = -10$
17. $-\frac{7}{8}x = -y + 5$
18. $2x = 9 + 3y$

19. From your results in Exercises 7–18, which way does a line slant if its slope is positive? if its slope is negative? Can you draw a line whose slope is zero?

B 20. Draw the graphs of the following three equations on the same set of axes.

a. $y = \frac{1}{2}x$ b. $y = 2x$ c. $y = 3x$

If you have graphed the lines correctly, they should intersect in one common point. What is that point?

— Graph the following to find the point that is common to each of the groups of three equations.

21. $y = 2$
$y = -x + 5$
$y = \frac{1}{3}x + 1$

22. $y = -2x + 1$
$y = -\frac{2}{5}x + 2\frac{3}{5}$
$y = \frac{1}{2}x + 3\frac{1}{2}$

23. $x - y = 0$
$x + 4y = -5$
$3x + y = -4$

24. $y = 1\frac{1}{2}x - 8$
$2x + \frac{1}{2}y = 7$
$3x + 4y = 4$

25. $x + y = 1$
$y = x + 1$
$x = 0$

26. $y = 2x - 3$
$x - 6 = -y$
$4y - x = 9$

27. $y = -\frac{3}{2}x + 4$
$y = -2$
$y = x - 6$

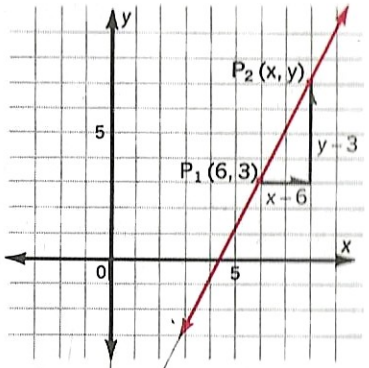
28. $2x + y = -10$
$3y + 10 = -x$
$y = \frac{1}{2}x$

4–8 Writing Equations for Lines

Given the slope of a line, m, and the coordinates of the y intercept, $(0, b)$, you can write the equation of the line in the form $y = mx + b$. Now you will learn some other methods for writing equations.

Equations from a Point and the Slope

Suppose you are given the coordinates (6, 3) and the slope 2, and you are asked to determine the equation of a line through the given point and having the given slope.



① Let (x, y) be the coordinates of any other point on the line containing (6, 3). Then the change in y is $y - 3$ and the change in x is $x - 6$. The slope of the line may be expressed as $\frac{y-3}{x-6}$.

But the slope is given as 2.

$$\frac{y-3}{x-6} = 2$$

$$(x-6) \cdot \frac{y-3}{x-6} = 2(x-6)$$

$$y - 3 = 2x - 12$$

$$y = 2x - 9$$

This is the equation of the line.

② Check to see that (6, 3) is a point on the line.

$3 = 2(6) - 9$, or $12 - 9$, or 3

Suppose you are given the **point** (x_1, y_1) and the **slope** m. Then using the method above, the following is true.

$$\frac{y - y_1}{x - x_1} = m$$

Multiply both sides of the equation by $(x - x_1)$.

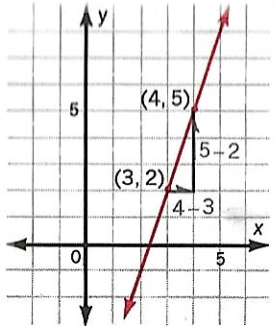
$$(x - x_1)\left(\frac{y - y_1}{x - x_1}\right) = m(x - x_1)$$

$$y - y_1 = m(x - x_1)$$

This is the **point-slope form** of the equation of a line and can be used to find the equation of a line, given any point and slope.

Equations from Two Points

Suppose you are given the coordinates for two points, (3, 2) and (4, 5), and you are asked to determine the equation of the line passing through these two points.



① Find the slope of the line through the points.

$$m = \frac{5-2}{4-3} = \frac{3}{1}, \text{ or } 3$$

Then in $y = mx + b$ form, you can write $y = 3x + b$. You must still determine b.

② If (3, 2) and (4, 5) are on the line for the desired equation, then they must satisfy the equation. Substitute the coordinates of either of the points in the equation to find b. Use (3, 2).

$$\begin{aligned} y &= 3x + b \\ 2 &= 3(3) + b \\ 2 &= 9 + b \\ -7 &= b \end{aligned}$$

Then the equation is $y = 3x - 7$.

③ Use the coordinates of the second point (4, 5) to check.

$$\begin{aligned} y &= 3x - 7 \\ 5 &= 3(4) - 7 \\ 5 &= 12 - 7, \text{ or } 5 \end{aligned}$$

If the two points were given as (x_1, y_1) and (x_2, y_2), you could follow the same method to find the equation of a line.

① Find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Then using $y = mx + b$, you get

$$y = x\left(\frac{y_2 - y_1}{x_2 - x_1}\right) + b.$$

② Substitute the coordinates of one of the points, say (x_1, y_1), in the equation and get b.

$$y_1 = x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right) + b$$

$$y_1 - x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = b$$

② Substitute the value for b in the equation from step ① and simplify.

$$y = x\left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1 - x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$y - y_1 = x\left(\frac{y_2 - y_1}{x_2 - x_1}\right) - x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right), \text{ or } (x - x_1)\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

This is the **two-point form** of the equation of a line and can be used to find the equation of a line through any two given points.

Now try these

Write an equation of a line, using the information given.

1. The slope is −2; a point is (−3, 1).
2. One point is (0, 1); another point is (2, −4).
3. $m = \frac{3}{4}$, $(x, y) = (2, 0)$

Answers: 1. $y = -2x - 5$ 2. $y = -\frac{5}{2}x + 1$ 3. $y = \frac{3}{4}x - 1\frac{1}{2}$

Checkpoint

How do you find the equation of a line, using the point-slope form? using the two-point form?

Exercises

A Given the following points and slopes, find the equations by the point-slope method. Write the equations in $y = mx + b$ form.

1. (2, 3); $m = 2$
2. (0, 4); $m = 3$
3. (4, 5); $m = -3$
4. (0, 0); $m = 1$
5. (−2, 1); $m = 1$
6. (0, 0); $m = -1$
7. (−1, 1); $m = \frac{1}{2}$
8. $(\frac{1}{2}, 3)$; $m = -\frac{2}{3}$
9. (−3, 4); $m = -\frac{1}{3}$
10. $(\frac{1}{4}, \frac{1}{5})$; $m = -1$

Find the equation of the line passing through the given points.

11. (1, 1), (2, 3)
12. (0, 0), (−3, 4)
13. (0, 1), (−1, 1)
14. (0, 0), (−1, −2)
15. (3, 4), (−2, 5)
16. (5, −1), (3, −2)
17. (−3, −4), (−5, −6)
18. (2, 3), (−1, 5)
19. (4, −3), (6, 0)
20. (1, 0), (0, 0)

B — In each of the following exercises, use the given slope and one point to write an equation of a line. Then determine whether the other point lies on that line.

21. 6, (2, 3); (6, 9)

22. $\frac{1}{2}$, (4, 2); (−2, −1)

23. $-\frac{3}{5}$, (−5, 7); (0, 4)

24. −4, (2, 6); (−1, 13)

25. 2, (−1, −2); (2, 4)

26. −3, (−2, 2); (1, 6)

— Tell whether or not the three points in each of the following exercises are collinear.

27. (−5, 11), (0, 8), (5, 5)

28. (0, 1), (4, −2), (6, −4)

29. (6, 3), (3, 2), (0, 0)

30. (4, 1), (−1, 7), (3, 3)

4–9 Direct Variation

Consider the simple relationship between the total cost and the number of six-cent stamps you buy. If c represents the total cost and n the number of stamps, the relation can be shown in the following ways.

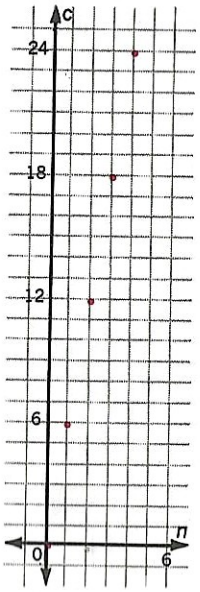
① A sentence: The total cost of n 6-cent stamps is $6n$.

② An equation: $c = 6n$.

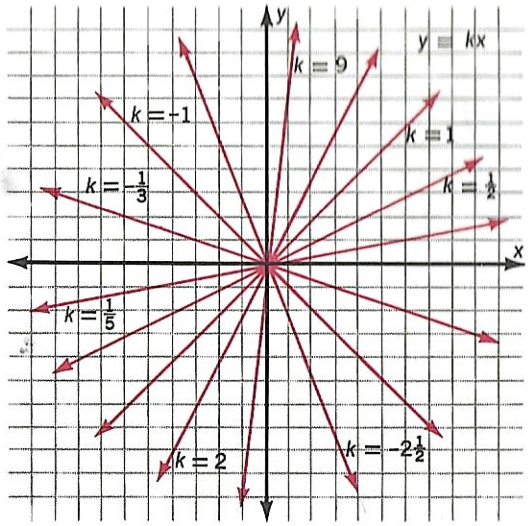
③ A table of ordered pairs.

n	0	1	2	3	4
c	0	6	12	18	24

④ A graph, as shown at the right.



For any x and y, if y is always equal to the product of x and a constant k, $y = kx$, then y is said to *vary directly* as x. The number that k represents is called the *constant of variation*. Examine the following diagram to see the effect the constant has on the graphs.



In the postage-stamp example, c is said to vary directly as n; that is, c is always 6 times as large as n.

No matter which way is chosen to show the relation, the important idea is that the first number, c, is always a multiple of a constant and the second number n. This illustrates **direct variation,** a "times-as-much" relation.

When $y = kx$ is true, where k is a constant number, there is a direct variation between x and y.

Any equation of the form $y = mx + b$, $m \neq 0$, is a linear equation. The equation $y = mx$, $b = 0$, shows *direct variation* between x and y.

You can contrast direct variation with an example such as the cost of repairing a television set. The repairman charges \$10 for a house call and \$6 per hour. Then the total cost, c, is 6 times the number of hours, h, plus \$10, that is, $c = 6h + 10$. If h is one hour, then c is 16; if h is 2, then c is 22; if h is 3, then c is 28. The relation in this example is not one of direct variation because it is not a simple case of multiplication only. Special attention is given to relations that are direct variations because they have such wide applications.

Checkpoint

1. Explain *direct variation.*
2. What is a constant of variation?

Exercises

A **1.** Write a formula that expresses the relation between the perimeter, P, of a square and the length, s, of a side. Is this an example of direct variation? Explain.

2. John earns \$1.25 an hour mowing lawns. Write an equation that expresses the relation between the amount earned in cents, A, and the number of hours worked, n. Is this an example of direct variation? Explain.

3. Look at the following table.

x	0	1	2	3	4	5	6
y	0	$2\frac{1}{2}$	5	$7\frac{1}{2}$	10	$12\frac{1}{2}$	15

a. Does each number for y equal a constant times the corresponding number for x? Does y vary directly as x? Explain.

b. Write an equation that shows the relation between x and y. ($y =$?)

c. When x is 4, y is 10. When x is 8, y is ?. If you double the value of x, is the corresponding value of y doubled? If you choose any value of x and double it, the corresponding value of y is ?.

d. When x is 2, y is ?. When x is 6, y is ?. If you triple any value of x, the corresponding value of y is ?.

4. Which of the following are examples of direct variation?

a. $x = 4y$	**b.** $x + y = 6$	**c.** $y = 6x$
d. $x - y = 4$	**e.** $\frac{x}{4} = y$	**f.** $y = \frac{x}{6}$
g. $x = -5y$	**h.** $x + 5y = 10$	**i.** $x - 8y = 0$
j. $xy = 10$	**k.** $xy + 4 = y$	**l.** $y = \frac{1}{4}x$
m. $\frac{x}{y} = 7$	**n.** $y = \frac{18}{x}$	**o.** $y - 5x = 9$

5. The equation $y = \frac{1}{2}x$ shows that y varies directly as x.

a. When x is 4, y is ?. When x is 8, y is ?. If you double any value of x, the corresponding value of y is ?.

b. When x is 1, y is ?. When x is 3, y is ?. If you triple any value of x, the corresponding value of y is ?.

c. When x is 10, y is ?. When x is 5, y is ?. If you multiply any value of x by $\frac{1}{2}$, the corresponding value of y is ?.

d. Is the graph of the equation a line if $x \in \mathbb{R}$?

6. $C = \pi d$ expresses the relation between the circumference of a circle and the length of a diameter.

a. Does C vary directly as d? Explain.

b. When d is $10''$, C is $10\pi''$. When d is $20''$, C is ?. If you take one circle and draw another circle whose diameter is twice that of the first, its circumference is ? the circumference of the first circle.

7. If y varies directly as x, and you choose a number for x and double it, is the corresponding number for y doubled? If you choose a number for x and triple it, the corresponding number for y is ?. If one number for x is multiplied by n, the corresponding number for y is multiplied by ?.

8. The equation $y = -5x$ shows that y varies directly as x.

a. Make a table showing at least six number pairs that satisfy the equation.

b. Pick a number for x and double it. Is the corresponding number for y doubled?

c. As x increases, y ? (increases or decreases).

d. If one number for x is halved, the corresponding number for y is ?.

e. If any number for x is multiplied by n, the corresponding number for y is ?.

9. If y equals a positive number times x, then

a. as x increases, y ? (increases or decreases).

b. as x decreases, y ? (increases or decreases).

c. when any number for x is doubled, the corresponding number for y is ?.

d. when any number for x is multiplied by n, the corresponding number for y is multiplied by ?.

a. as x increases, y ? (increases or decreases).

b. as x decreases, y ? (increases or decreases).

c. when any number for x is doubled, the corresponding number for y is ?.

d. when any number for x is multiplied by n, the corresponding number for y is ?.

11. If y varies directly as x, which statements are always true?

a. If x increases, y increases.

b. If a value of x is multiplied by n, the corresponding value of y is multiplied by n.

c. The graph of the relation between x and y is a line.

B — Tell which of the following exercises illustrate direct variation. Give an example to verify your answer.

12. The perimeter of an equilateral triangle varies with respect to one side.

13. The amount of interest at a fixed rate earned on a given sum of money varies with respect to time.

14. The length of a rectangle of a fixed perimeter varies with respect to the width.

15. A steel bar's length varies with respect to its temperature.

4–10 Direct Variation and Proportions

You know that $y = kx$ expresses direct variation between x and y. In chemistry and in other areas of science, it is often easier to express direct variation by using a **proportion.**

Consider (x_1, y_1) and (x_2, y_2). If $\frac{y_1}{x_1} = 5$ and $\frac{y_2}{x_2} = 5$ are true, then $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ is true because each term in the equation is equal to 5. The equation $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ is a proportion.

Definition A proportion is an equation that shows that the ratio of two numbers is equal to the ratio of two other numbers.

Think of $y = kx$ expressed as the equivalent equation $\frac{y}{x} = k$. The ratio of y to x is always a constant k. If k is 5, then any given value of y divided by the corresponding value of x is 5. Then you would know that any two number pairs that satisfy the equation $\frac{x}{y} = 5$ would have the same ratio; (1, 5) and (2, 10) are two such pairs, since $\frac{5}{1}$ and $\frac{10}{2}$ are both names for 5.

EXAMPLE 1. A recipe uses 3 cups of sugar for each 24 cookies. How much sugar is needed for 66 cookies?

You can summarize the information as follows to set up the proportion.

24 cookies	66 cookies
3 cups of sugar	x cups of sugar

Since the problem shows direct variation, you can write the following proportion.

$$\frac{24}{3} = \frac{66}{x}$$

Then solve the proportion for x.

$$\frac{24}{3}, \text{ or } \frac{8}{1} = \frac{66}{x}$$
$$\frac{8}{1} \cdot x = \frac{66}{x} \cdot x$$
$$8x = 66$$
$$x = \frac{66}{8}, \text{ or } 8\tfrac{1}{4}$$

Thus, $8\frac{1}{4}$ cups of sugar are needed for 66 cookies.

Often, you will see the term *directly proportional* refer to direct variation when it is expressed as a proportion.

Checking to see that two ratios are equal, or that a proportion is true, is done in the same way as for two rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$.

If $\frac{a}{b} = \frac{c}{d}$ is true, then $ad = bc$ is true.

$$\frac{a}{b} = \frac{c}{d}$$
$$bd\left(\frac{a}{b}\right) = bd\left(\frac{c}{d}\right)$$
$$da = bc$$
$$ad = bc$$

The definition for equality of two ratios can be used to solve a proportion such as the one in Example 1, or it may be used to check the result.

Solve Example 1 again.

$$\frac{24}{3} = \frac{66}{x}$$
$$24x = 3 \cdot 66, \text{ or } 198$$
$$x = \tfrac{198}{24}, \text{ or } 8\tfrac{1}{4}$$

Check

$$\frac{24}{3} = \frac{66}{8\frac{1}{4}}$$
$$24 \cdot 8\tfrac{1}{4} = 3 \cdot 66$$
$$198 = 198$$

EXAMPLE 2. Solve the proportion $\frac{x}{13} = \frac{25}{65}$ for x.

$$\frac{x}{13} = \frac{25}{65}$$
$$65x = 13 \cdot 25$$
$$65x = 325$$
$$x = \tfrac{325}{65}, \text{ or } 5$$

Check

$$\frac{5}{13} = \frac{25}{65}$$
$$325 = 325$$

Now try these

— Solve the following proportions.

1. $\frac{x}{2} = \frac{9}{18}$
2. $\frac{2}{3} = \frac{x}{27}$
3. $\frac{5}{x} = \frac{20}{48}$
4. $\frac{9}{11} = \frac{16}{x}$

— Write the proportion you would use to solve these problems. You need not solve them at this time.

5. If you can travel for 120 miles and use 6 gallons of gasoline, how far can you travel at the same rate of gas consumption and use 15 gallons of gasoline?
6. If 36 grams of salt will dissolve in 100 grams of water at a given temperature, how many grams of salt will dissolve in 250 grams of water at the same temperature?

Answers: 1. 1 2. 18 3. 12 4. $19\frac{5}{9}$ 5. $\frac{6}{120} = \frac{15}{x}$ 6. $\frac{36}{100} = \frac{x}{250}$

Checkpoint

1. What is a *proportion*?
2. How do you solve a proportion?

Exercises

A — Solve for x.

1. $\frac{7}{8} = \frac{x}{40}$
2. $\frac{9}{2} = \frac{63}{x}$
3. $\frac{11}{x} = \frac{132}{24}$
4. $\frac{x}{13} = \frac{10}{65}$
5. $\frac{7}{x} = \frac{1}{9}$
6. $\frac{5}{13} = \frac{x}{65}$
7. $\frac{21}{5} = \frac{x}{2.5}$
8. $\frac{29}{7} = \frac{x}{1.75}$
9. $\frac{17}{2} = \frac{8.5}{x}$

— Solve for n.

Example. $\frac{2+n}{n} = \frac{15}{9}$

$$9(2+n) = 15n$$
$$18 + 9n = 15n$$
$$18 = 6n$$
$$3 = n$$

Check

$$\frac{2+3}{3} = \frac{15}{9}$$
$$\tfrac{5}{3} = \tfrac{15}{9}$$
$$45 = 45$$

10. $\frac{12-n}{n} = \frac{5}{7}$
11. $\frac{n}{16-n} = \frac{5}{3}$
12. $\frac{n+4}{5} = \frac{2n}{6}$
13. $\frac{3n}{8} = \frac{6n}{16}$
14. $\frac{3+n}{n} = \frac{4}{3}$
15. $\frac{n}{3-n} = \frac{a}{b}$

16. If an automobile goes 110 miles in 3 hours, how far will it go in 5 hours at the same rate? $\left(\frac{110}{x} = \frac{3}{5}\right)$
17. If 90 feet of wire weighs 18 pounds, what will 110 feet of the same kind of wire weigh?
18. A recipe for a 2-pound cake uses $1\frac{1}{2}$ cups of butter. How many cups of butter are needed for a 5-pound cake?
19. If 40 pounds of fertilizer are used for 5000 square feet of lawn, how much is needed for the Clarks' lawn, which is rectangular in shape and measures 80 feet by 150 feet?
20. If $1\frac{1}{2}$ inches on a map represents 60 miles, what distance does $2\frac{7}{8}$ inches on the map represent?

B 21. Separate 72 into two parts that are in the ratio 4 to 5.

METHOD I: Let the smaller part be x. The other will be $72 - x$.

$$\frac{x}{72-x} = \frac{4}{5}$$

METHOD II: Let the smaller part be $4x$. The other will be $5x$.

$$4x + 5x = 72$$

Use either method to find the two parts.

23. Henry and his younger brother decided to divide a profit of \$12 in the ratio 5 : 3. How should it be divided?

24. If a man reaches the moon, he will weigh much less there than on earth. The ratio of the weight of an object on the moon to the weight of an object on earth is 0.16 to 1.

a. James Explorer weighs 150 pounds. How much would he weigh on the moon?

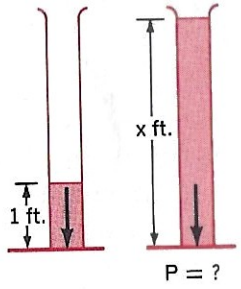
b. How much would a 10-ton rocket weigh on the moon?

c. A 100-pound sack of flour on the moon would weigh how much on earth?

C **25.** The heights of two columns of water are directly proportional to the pressures exerted by the columns. Since the pressure exerted by a column of water one foot high is 0.433 pound per square inch, this proportion can be written as follows.

$$\frac{1}{\begin{array}{l}\text{height of any}\\ \text{column of water}\\ \text{in feet}\end{array}} = \frac{0.433}{\begin{array}{l}\text{pressure of the}\\ \text{column in lb.}\\ \text{per sq. in.}\end{array}},$$

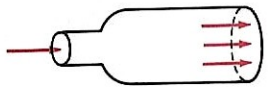
or $\frac{1}{x} = \frac{0.433}{P}$



What then is the pressure exerted by a column of water 20 feet high?

26. What is the water pressure (in pounds per square inch) on the bottom of a tank in which the water is 30 feet deep?

27. For a liquid contained in a closed pipe or container, the forces on any two areas are directly proportional to the areas. For example, for a bottle such as the one pictured at the right, this law may be expressed as follows.



$$\frac{\text{force at small end}}{\text{force at large end}} = \frac{\text{area of small end}}{\text{area of large end}}$$

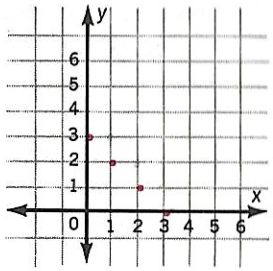
If the force at the small end is 50 pounds and its area is 12 square inches, what will be the force at the large end if its area is 108 square inches?

28. The principle in Exercise 27 governs the operation of hydraulic brakes. If a driver presses with a force of 10 pounds on a brake pedal with a hydraulic fluid line that has an area of 3 square inches, how much force will be applied on the piston at the brake if the piston has an area of 16 square inches?

4–11 Graphing Inequalities

You graphed a simple inequality such as $x > 3$ on a number line like that below, and you graph $x + y > 3$ on a plane.

To graph $x + y > 3$, you begin with the graph of $x + y = 3$. Suppose the replacement set for x and y is $\{0, 1, 2, 3, 4, 5, 6\}$. Four pairs of numbers make $x + y = 3$ true. These pairs are graphed at the right.



In relation to the graph of $x + y = 3$, where are the points whose coordinates satisfy $x + y > 3$?

For the inequality $x + y > 3$, if x is 2, y can be 2, 3, 4, 5, or 6. Where are the points for (2, 2), (2, 3), (2, 4), (2, 5), and (2, 6)?

Where are the points whose ordered pairs would have 1 as the x coordinate and still satisfy the inequality?

Since the replacement set is finite, you can list all the ordered pairs that satisfy the inequality. Complete the following table.

When x is	y can be
0	4, 5, 6
1	3, ?, ?, ?
2	2, 3, 4, 5, 6
3	1, ?, ?, ?, ?, ?
4	0, ?, ?, ?, ?, ?, ?
5	0, ?, ?, ?, ?, ?, ?
6	0, ?, ?, ?, ?, ?, ?

above and to the right of the points for $x + y = 3$, as in Figure 1.

Examine the points on the other side of the graph of $x + y = 3$. List the ordered pairs. Do the coordinates satisfy $x + y < 3$?

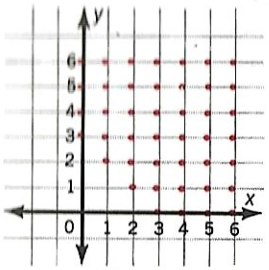


Figure 1

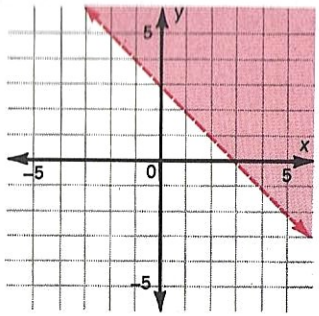


Figure 2

What is the graph of $x + y = 3$ if the replacement set is the set of real numbers? It is the dashed line shown in red on the graph in Figure 2. Then the graph of $x + y > 3$ would be the half-plane on the right side of the line for $x + y = 3$. The half-plane is shown by the shading above the red dashed line.

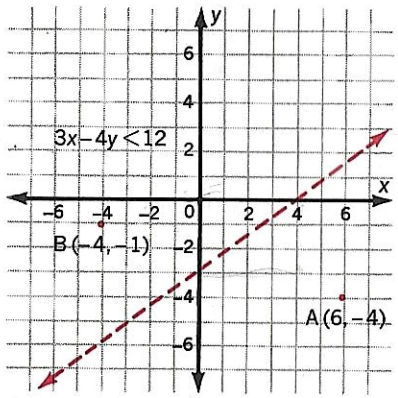
To graph an inequality whose replacement set is the set of real numbers, first graph the corresponding equation. Then points on one side of the line have coordinates that satisfy the inequality.

EXAMPLE. Graph $3x - 4y < 12$.

① Graph the equation

$$3x - 4y = 12.$$

Use a dashed line to represent the graph as shown at the right.



② Points on one side of the line have coordinates that satisfy $3x - 4y < 12$. Choose a point on one side of the line and see if its coordinates satisfy the inequality. Do the coordinates of A satisfy $3x - 4y < 12$?

$$3(6) - 4(-4) \stackrel{?}{<} 12$$
$$18 + 16 \stackrel{?}{<} 12$$
$$34 \not< 12$$

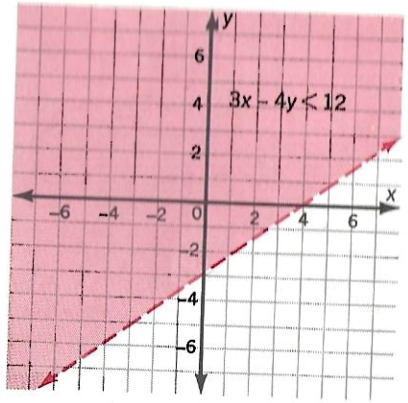
A is not on the graph of $3x - 4y < 12$.

Do the coordinates of B satisfy the inequality?

$$3(-4) - 4(-1) \stackrel{?}{<} 12$$
$$-12 + 4 < 12$$
$$-8 < 12$$

Then B is on the graph of $3x - 4y < 12$.

③ All points on the B side of the line have coordinates that satisfy the inequality. Shade this side of the line, as shown at the right. The half-plane that is the graph of $3x - 4y < 12$ extends indefinitely upward to the left. Why does it not include the dashed line?



The inequality $3x - 4y < 12$ is a **linear inequality** because the boundary of its graph is a line.

The relation defined by $3x - 4y < 12$ is not a function. Why?

Exercises

A — The replacement set is $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Graph the equation in **a**. Then, on the same axes, graph the inequalities in **b** and **c**.

	a	b	c
1.	$y = x + 8$	$y > x + 8$	$y < x + 8$
2.	$x - y = 6$	$x - y > 6$	$x - y < 6$
3.	$2x - 3y = 6$	$2x - 3y > 6$	$2x - 3y < 6$
4.	$3x + 4y = 12$	$3x + 4y > 12$	$3x + 4y < 12$
5.	$x = 8$	$x > 8$	$x < 8$
6.	$y = 7$	$y > 7$	$y < 7$

— Let the replacement set be the set of real numbers. Graph the following as you did Exercises 1–6.

7.	$2x - y = -4$	$2x - y > -4$	$2x - y < -4$
8.	$x = -3\frac{1}{2}$	$x > -3\frac{1}{2}$	$x < -3\frac{1}{2}$
9.	$\frac{1}{2}x + y = 4$	$\frac{1}{2}x + y > 4$	$\frac{1}{2}x + y < 4$
10.	$3x - 2 = 5y$	$3x - 2 > 5y$	$3x - 2 < 5y$

— Graph each inequality. The replacement set is the set of real numbers.

11. $x + y > 10$	12. $x - y > 10$	13. $y - x > 10$
14. $2x + y > 5$	15. $x - 2y < 4$	16. $2x - y > -4$
17. $3x - 2y < -6$	18. $y > -5$	19. $x > 0$

— Graph the following. The replacement set is the set of real numbers.

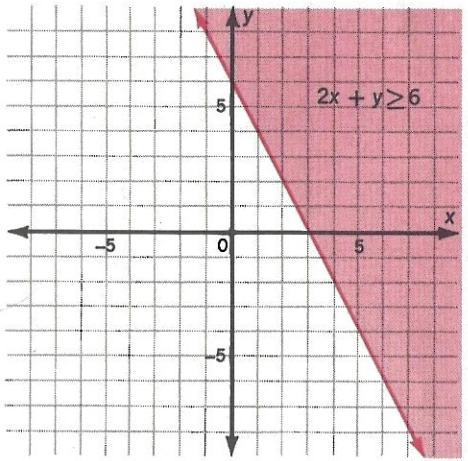
Example. Graph $2x + y \geq 6$.

Recall that $\geq$ means "is greater than or equal to" and $\leq$ means "is less than or equal to."

The graph for $2x + y \geq 6$ is the union of the graphs for

$$2x + y > 6 \text{ and } 2x + y = 6.$$

This includes the half-plane and the line.



20. $x + 2y \geq -5$	21. $x - y \leq 10$
22. $4x - 3y \geq 12$	23. $x \geq -5$
24. $y \leq -9$	25. $\frac{5}{4}x - y \leq 8$
26. $x \geq 0$	27. $y \leq 0$

B — The replacement set for the following is the set of real numbers. Graph each set.

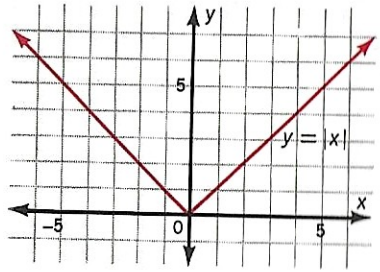
28. $\{(x, y) : x + 3y \geq 2\} \cup \{(x, y) : x > 0\}$
29. $\{(x, y) : -2x - y \leq 4\} \cap \{(x, y) : x = y\}$
30. $\{(x, y) : 5x - 4y \geq 2\} \cap \{(x, y) : y \geq 0\}$
31. $\{(x, y) : 3x + y \leq 7\} \cup \{(x, y) : x - y = 3\}$

4–12 Absolute Value — Relations and Functions

To graph an equation such as $y = |x|$, find several pairs of numbers that satisfy the equation and locate the points. Recall that x may be positive, negative, or zero, but y will always be nonnegative, by definition of absolute value. The replacement set is the set of real numbers. Complete the table of pairs of numbers that satisfy $y = |x|$.

When x is	−5	−4	−3	−2	−1	0	1	2	3	4	5
y is	5	?	?	2	?	0	1	2	3	?	?

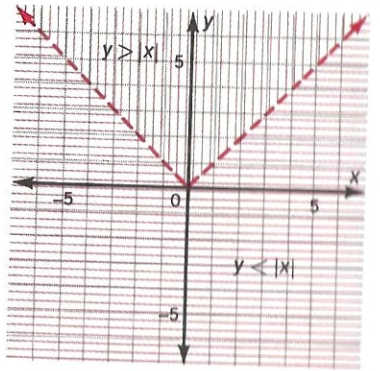
Locate a point for each pair of numbers. All points whose coordinates satisfy the equation will form a V-shaped figure, that is, two rays with their initial point at the origin.



The relation defined by $y = |x|$ is a function. Explain why this is so. Why is it not a linear function? How does the graph of $y = |x|$ compare with the graph of $y = x$?

To graph $y > |x|$, check the coordinates of points inside (or above) the V. Try (0, 4). Is $4 > |0|$ true? Try (2, 5). Is $5 > |2|$ true? Try (−4, 5). Is $5 > |-4|$ true?

Check by substitution to see that points outside the V satisfy $y < |x|$. The graphs are shown at the right.



Neither of the relations defined by $y > |x|$ and $y < |x|$ is a function. Can you explain why?

Exercises

A — Graph these equations if the domain is the set of real numbers. If the equation defines a function, write the word *Function*.

1. $y = \|x\| - 4$	2. $y = \|2x\| - 5$
3. $y = \|-3x\| + 7$	4. $y = \|5x\|$
5. $y = -\|x\|$	6. $y = -\|2x\|$

(*Hint:* In Exercises 5 and 6, every value of y will be negative except when $x = 0$.)

7. $x = |y|$ **8.** $x = |2y| - 4$

9. $x = |-2y|$ **10.** $x = |-4y| + 3$

11. $x = -|3y|$ **12.** $x = -|2y| - 6$

13. $x + |y| = 4$ **14.** $x - |y| = 6$

15. $y + |x| = 10$ **16.** $|x| - y = 8$

B — Graph the following equations. The replacement set is the set of real numbers.

17. $|x| + |y| = 10$ (*Hint:* Neither x nor y can be greater than 10 or less than -10. Why?)

x	-10	$+10$	-8	$+8$	-6	$+6$	-4	...
y	0	0	± 2	± 2	?	?	?	...

Continue the table for a few more values of x and draw the graph. What kind of figure do you obtain?

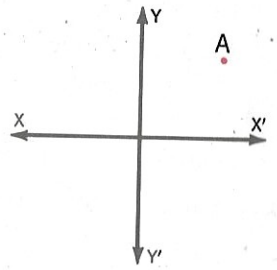
18. $|y| = |x| + 5$ **19.** $|x| = |y|$

CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words, phrases, or equations. The number shown after each word, phrase, or equation indicates where it is introduced, in case you need to review.

undefined term (*162*)
point (*162*)
line (*162*)
line segment (*162*)
endpoints (*162*)
ray (*162*)
angle (*163*)
half-line (*163*)
plane (*163*)
half-plane (*164*)
boundary (*164*)
polygon (*164*)
interior (*164*)
exterior (*164*)
parallel lines (*164*)
perpendicular lines (*164*)
quadrant (*166*)
ordered pair (*168*)
axes (*169*)
origin (*169*)
abscissa (*169*)
ordinate (*169*)
coordinates (*169*)
coordinate plane (*169*)
rectangular-coordinate plane (*169*)
polar-coordinate plane (*173*)
polar axis (*173*)
relation (*174*)
domain (*175*)
range (*175*)
function (*178*)
independent variable (*178*)
dependent variable (*178*)
linear function (*179*)
$y = mx + b$ (*182*)
slope (*186*)
y intercept (*187*)
$y - y_1 = m(x - x_1)$ (*193*)
$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ (*195*)
direct variation (*197*)
proportion (*200*)

2. Consider the figure at the right. Which of the following names the part of the plane containing point A?



a. X' side of $\overleftrightarrow{YY'} \cap$ Y' side of $\overleftrightarrow{XX'}$

b. Y' side of $\overleftrightarrow{XX'} \cap$ X side of $\overleftrightarrow{YY'}$

c. X side of $\overleftrightarrow{YY'} \cap$ Y side of $\overleftrightarrow{XX'}$

d. X' side of $\overleftrightarrow{YY'} \cap$ Y side of $\overleftrightarrow{XX'}$

3. Draw coordinate axes on graph paper. Then draw a line connecting $(-6, -1)$, $(-3, 2)$, and $(0, 5)$ and a line connecting $(-3, -6)$, $(0, -3)$, and $(3, 0)$. What appears to be the relationship between the two lines?

4. It costs \$3.00 to print the first 100 copies of a handbill and \$2.00 for each 100 additional copies. This relation may be expressed by the equation $c = 3 + 2x$, where x represents the number of 100's of copies after the first 100. Express the relation as a table. The domain of x is $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Express the relation as a graph, using the ordered pairs in the table. Why is the graph a set of discrete points?

5. For each equation, give three ordered pairs that satisfy it. Draw the graph for each equation if the domain is the set of real numbers.

a. $3x - 2y = 5$ **b.** $-2x - 7y = 4$ **c.** $5x + y = 7$

6. Consider the equation $2x - 5y = 8$.

a. Is this a linear function? Explain.

b. Write the equation in the form $y = mx + b$.

c. What is the slope of the line that is the graph of the function?

d. What is the y intercept of the line?

e. Graph the line for $2x - 5y = 8$ if the replacement set is the set of real numbers.

7. A point on line l has coordinates $(0, -3)$. The slope of the line is $-\frac{3}{2}$. Write the equation of the line. Draw line l.

8. Two points on line m are $(-3, 4)$ and $(-4, -3)$. Write the equation of the line that contains these two points. Draw line m.

9. Which of the following equations are examples of direct variation? Why?

a. $xy = -3$ **b.** $\frac{x}{y} = 5$ **c.** $\frac{x}{y} = \frac{x'}{y'}$ **d.** $x = -\frac{2}{3}y$ **e.** $x - y = 0$

10. Which of the following is an example of direct variation? Why?

a. At 50 miles per hour, a driver may drive $50n$ miles in n hours.

b. On a business trip, Mr. Jacobs drove 15 miles in the city in heavy traffic and then drove at 50 miles per hour on the highway for h hours. The distance he drives in all is represented by the equation $d = 15 + 50h$.

11. Solve the following proportions for x.

a. $\frac{4}{x} = \frac{-2.5}{10}$ **b.** $\frac{x}{16} = \frac{2}{3}$ **c.** $\frac{9}{25} = \frac{x}{75}$

12. Alice receives a commission of 8% on the total amount she sells in a week. Find her commission, x, on \$525 by writing a proportion and solving for x. (*Hint:* Remember 8% may be written as $\frac{8}{100}$.)

13. A punch recipe calls for 1 cup of lemon soda for each 3 cups of orange juice. If you use 10 cups of orange juice, how many cups of soda will you need? Solve by a proportion.

14. Graph each inequality. Let the replacement set be the set of real numbers.

a. $3x - 2y < 12$ **b.** $2x - 5y \geq 9$

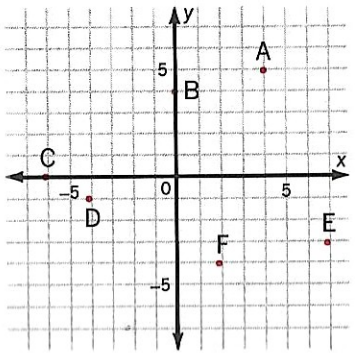
CHAPTER TEST

1. Write the coordinates for points A, B, C, D, E, and F.

2. If you graph all points such that $x < 0$, what kind of figure is the graph? Are all the points on one side of the x axis? on one side of the y axis? Does $x < 0$ define a relation that is a function or one that is not a function?

3. Which point in the figure is in the quadrant where $x < 0$ and $y < 0$?

4. Which labeled points are not in any of the four quadrants?



5. Without graphing, write *Yes* if the ordered pair is a member of the given set. Otherwise write *No*.

a. $\{(x, y) : 2x - y = 8\}$; $(4, 0)$

b. $\{(x, y) : 2x + 3y = 11\}$; $(-1, 3)$

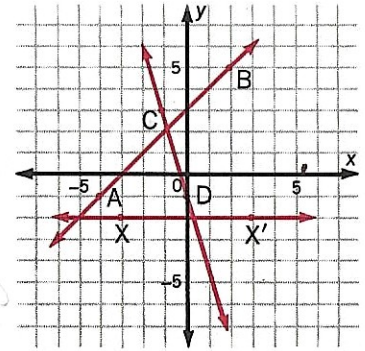
c. $\{(x, y) : |x| + |y| = 4\}$; $(-1, 3)$

d. $\{(x, y) : x > 3y - 9\}$; $(0, 3)$

e. $\{(x, y) : 3x - 2y \leq 16\}$; $(4, -2)$

f. $\{(x, y) : y = 5\}$; $(7, 5)$

6. What is the slope of $\overleftrightarrow{AB}$? $\overleftrightarrow{CD}$? $\overleftrightarrow{XX'}$?



7. What is the slope of the graph of these equations?

a. $y = 5x - 6$ **b.** $y = -\frac{2}{3}x - 9$

c. $x + 3y = 8$ **d.** $2x - 6y = 11$

8. Which equations in Exercise 7 define a function? a linear function?

9. Write the equation of the line passing through $(-5, 6)$ with a slope of 3; through $(2, 3)$ with a slope of $-\frac{1}{2}$.

Graph the following relations. Write *Function* if the sentence defines a function. Otherwise, write *Relation*.

10. $2x + y = 12$ **11.** $y > 3x + 5$

12. $y = |x| - 2$ **13.** $x + y \leq 4$

Graph the following.

14. $\{(x, y) : x + y = 10\} \cup \{(x, y) : x - y = 4\}$

15. $\{(x, y) : y \geq 3x\} \cap \{(x, y) : -3 \leq y \leq 3\}$

Solve for x.

16. $\frac{x}{7} = \frac{15}{21}$ **17.** $\frac{14}{x} = \frac{25}{13}$ **18.** $\frac{x}{7} = \frac{7}{8}$ **19.** $\frac{6}{18} = \frac{14}{x}$

20. Write the equation of the line passing through $(-6, 2)$ and $(4, -\frac{1}{2})$; through $(3, 3)$ and $(-1, -5)$.

21. Which of the following are examples of direct variation?

a. The length of one side of a square and the perimeter.

b. The number of men working on a job and the number of weeks required to complete it.

c. The altitude of a parallelogram of constant area and the base.

d. Temperature and the length of a column of mercury in a thermometer.

e. Temperature in an oven and the amount of time needed for something to cook.

22. The ratio of the areas of two squares is 7 : 4. If the length of a side of the first square is 16 inches, what is the length of a side of the second square?

23. The scale on a map is given as 1 inch = 100 miles. How great a distance is represented by a line $2\frac{3}{8}$ inches long on the map?

ENRICHMENT

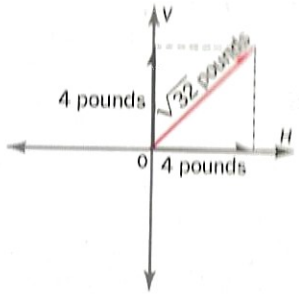
Vector and Force Problems

A force applied to a point can be thought of as a vector. A vector has magnitude, that is, a numerical value, and direction. A force has a numerical value, such as 10 pounds or 2 tons, and a direction in which it is acting, such as left, right, up, down, or at some acute angle.

Think of two unequal forces applied at one point, acting in opposite directions. An example of this would be a tug of war. One team pulls to the left with a force of 300 pounds; the other team pulls to the right with a force of 240 pounds. The forces can be represented by a vector diagram.

You know which team will win the tug of war — the team pulling with the greater force. How much greater is the force to the left? If you think of the 300-pound force as two forces, one of 240 pounds and one of 60 pounds, both acting in the same direction, then you can see that the 240-pound forces cancel each other out, and the **resultant force** is 60 pounds directed to the left.

What would be the result of two 4-pound forces acting at right angles to pull a rock? Represent the forces on coordinate axes with the rock at the origin. If you just think about it, you know that the rock would not move up because there is also a force pulling it to the right, and it would not move to the right because there is also a force pulling it up. Thus, it will move in some direction between the two.



The problem can be solved mathematically by completing a vector rectangle. The diagonal of the rectangle is the resultant vector and can be found by the Pythagorean theorem to be $\sqrt{4^2 + 4^2}$, or $\sqrt{32}$, pounds.

Similarly, if you know the magnitude and direction of any vector, you can separate it into its horizontal and vertical components by placing the initial point of the vector at the origin and completing a vector rectangle. Each vector can be named by a pair of real numbers, those numbers being the coordinates of the endpoint of the vector, its initial point being the origin. Look at vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ in Figure 1 below. The name for $\vec{a}$ is (2, 0); for $\vec{b}$ is (−3, 0); and for $\vec{c}$ is (4, 3). This number pair then tells you both the length and direction of the vector.

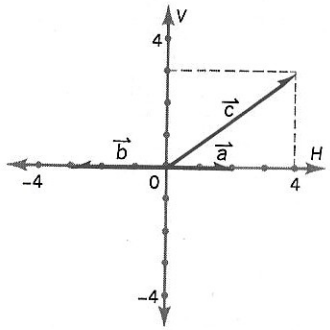


Figure 1

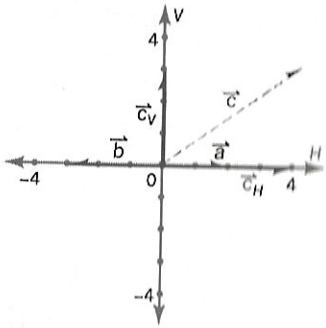
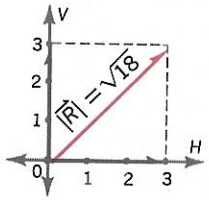


Figure 2

If you want the resultant of $\vec{a}$, $\vec{b}$, and $\vec{c}$, then $\vec{c}$ must be separated into its horizontal and vertical components, c_H and c_V, respectively. Look at Figure 2 above; $\vec{c_H}$ is (4, 0) and $\vec{c_V}$ is (0, 3). First, find the resultant of the horizontal forces: $-3 + 4 + 2 = +3$. Now you can find the resultant of two +3 vectors at right angles to each other by the Pythagorean theorem: $\sqrt{3^2 + 3^2} = \sqrt{18}$.

So the magnitude of the resultant of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is "$\sqrt{18}$ directed as shown in the figure below."



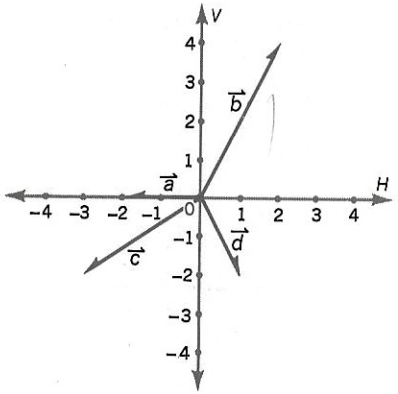
But it should be expressed in a way more precise than that. You can replace this geometric addition of vectors by real-number arithmetic. To add $\vec{a}$ (2, 0), $\vec{b}$ (−3, 0), and $\vec{c}$ (4, 3), add the coordinates in the following way.

$$(2, 0) \oplus (-3, 0) \oplus (4, 3) = (2 - 3 + 4, 0 + 0 + 3) = (3, 3)$$

Thus, $\vec{R}$ is (3, 3), which is the same as $\sqrt{18}$ in the previous diagram. Note that the sign $\oplus$ is used to alert you to the fact that you are defining a new type of addition.

Exercises

1. Copy the vector diagram at the right, substituting for each vector its horizontal and vertical components. Then find the resultant vector by combining the horizontal and vertical components and using the Pythagorean theorem.



2. Find the resultant of the vectors in Exercise 1 by vector addition, $\oplus$. Do your two answers check?

For each of the following, draw a vector diagram and find the resultant by vector addition, $\oplus$.

3. $\vec{a}$ (4, 1), $\vec{b}$ (0, 2), $\vec{c}$ (−2, 4), $\vec{d}$ (−3, 0), $\vec{e}$ (−2, −2), $\vec{f}$ (0, −3), $\vec{g}$ (3, −2)

4. $\vec{a}$ (−4, 0), $\vec{b}$ (−3, −4), $\vec{c}$ (−2, −4), $\vec{d}$ (0, −4), $\vec{e}$ (−3, 3)

5. An engineer had a problem involving forces and he was using vectors to solve it. He had finished the problem and gone for coffee when the custodian accidentally threw away one page of the solution because it contained only one little ordered pair. The information that the engineer had left was $\vec{a}$ (2, 5), $\vec{b}$ (−3, −2), $\vec{c}$ (0, 4), $\vec{d}$ (3, −4), $\vec{R}$ (7, 7). ($\vec{R}$ is the resultant.) What was $\vec{e}$, the vector that was thrown away?

6. Suppose in Exercise 5 that there had been two vectors on the paper that was thrown away. The engineer knew that the vectors he had been working with were all named by pairs of integers. And he remembered that the two lost vectors were both directed upward to the right, one having a magnitude of 5 pounds. Can you determine the two vectors?

THE GROWTH OF MATHEMATICS

The Pythagoreans

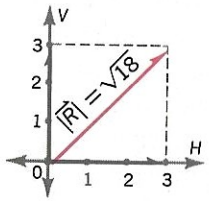
Geometry and deductive thinking were introduced into Greece by Thales (640?–546 B.C.). He traveled to Egypt as a trader and learned what he could from the priests there, returning to Miletus in Greece to study philosophy and mathematics.

A contemporary of Thales and, perhaps, a student of his was Pythagoras (569?–?500 B.C.). Born on the island of Samos, he too studied in Egypt and Babylonia. Eventually, he settled at Croton in southern Italy where he founded a secret brotherhood whose members were his students. Pythagoras was fascinated by numbers and came to see numbers in everything, believing that numbers ruled the universe. He attributed special properties to each natural number; for example, the odd numbers were associated with masculinity, even numbers with femininity; 1 was identified with reason, 2 with opinion, and so on.

Because of the secretive and mystical nature of the Pythagoreans, they were suspected by the politicians of Croton. Their buildings were attacked and destroyed, and Pythagoras was killed by a mob. The school regrouped itself, but some members dispersed and began to tell and write of the society's discoveries, always crediting everything to Pythagoras. Best known is the Pythagorean theorem. Their discovery of the irrationality of $\sqrt{2}$ was disturbing as it could not be reconciled with the whole numbers, the basis of their entire philosophy.

They contributed greatly to geometric algebra and even gave the complete solution of the general quadratic equation with real roots. They developed a rather complete theory of proportion and did much great work in geometry, refining deduction and proof to a high degree.

So the magnitude of the resultant of $\vec{a}$, $\vec{b}$, and $\vec{c}$ is "$\sqrt{18}$ directed as shown in the figure below."



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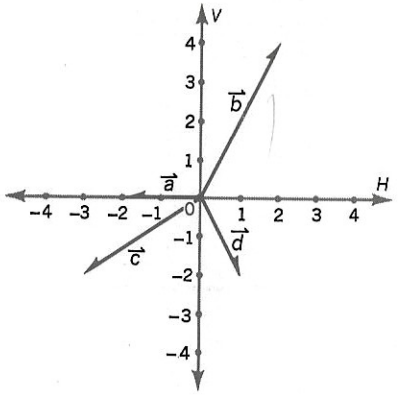

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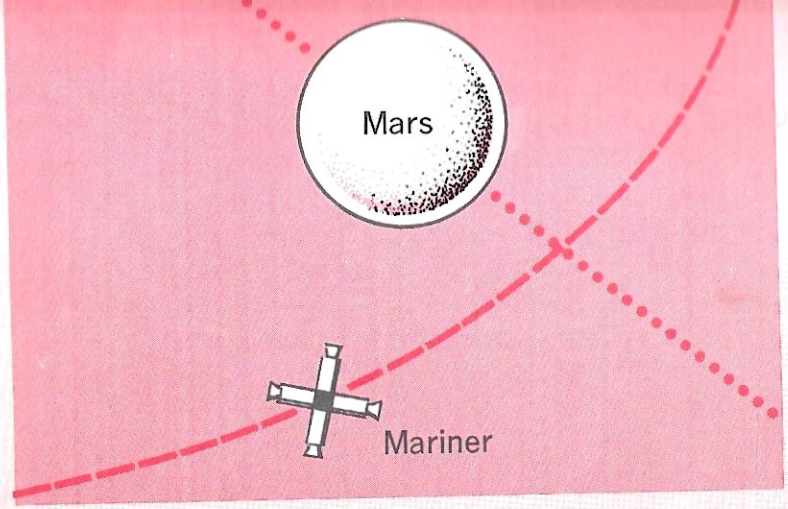
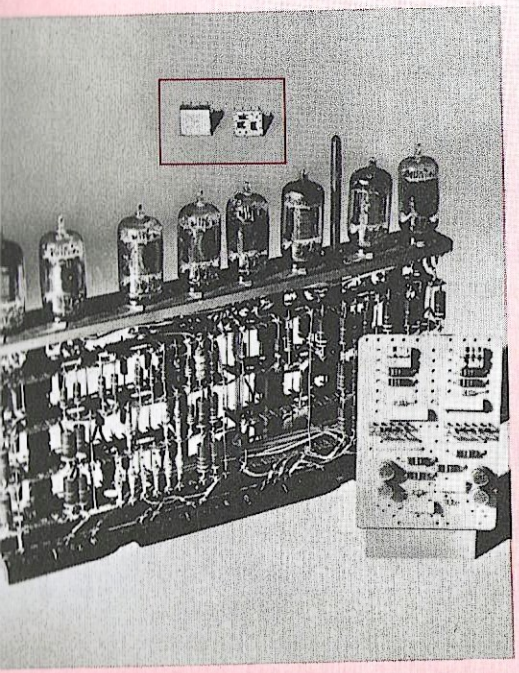
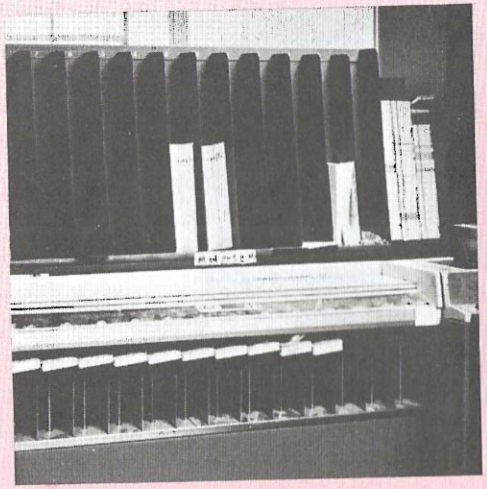






CHAPTER 5

SYSTEMS OF SENTENCES

You have been accustomed to solving problems by using a single equation or inequality. Many problems, however, must be described in terms of two or more equations or inequalities. Such systems of sentences can be solved through the use of graphs or by application of the postulates and theorems of algebra.

The photographs and diagrams suggest some ways to solve such systems.

- The diagram in the top corner shows the problem of hitting a moving planet with a moving space probe. Unless the probe is in the right part of its orbit when the planet arrives, the probe will miss. The point of intersection of the two paths is the point at which you must aim.
- The machine in the center right photograph sorts information on punched cards. You can use a card sorter to eliminate part of a problem so that the remaining part is easy to solve. A similar elimination process is used with systems of equations.
- The small crystal in the center left photograph is an integrated circuit that does the same work as the complicated electronic equipment also shown. Substitution of an integrated circuit can greatly simplify the construction of a television set or computer. In solving systems of equations, the substitution of the right expression can give the solution.
- In baseball, the rules state that the "fair ball" territory is between the rays from home plate through first and third base or on these rays. It is the region not shaded in the bottom photograph. As you will see in this chapter, this definition of "fair" territory is similar to the solution set of a system of two inequalities.

5–1 Graphing Systems of Sentences

Can you solve the following puzzle about a pair of numbers? Try it before reading the explanation.

"I am thinking of two numbers in a certain order. I will tell you two things about the numbers.

"① The sum of the two numbers is 7.

"② Multiply the first number by 3 and subtract 1, and you will get the second number.

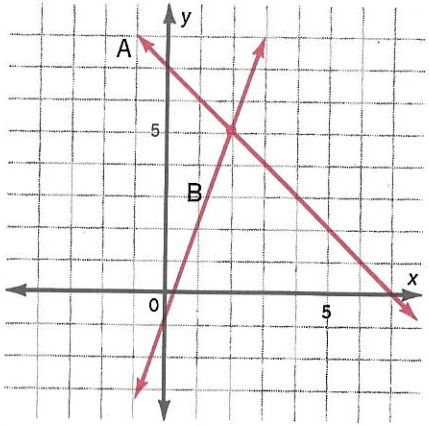
"What are the numbers? Which is the first number and which is the second number?"

The two conditions of the puzzle can be expressed by two equations in two variables. Letting x represent the first number and y the second number, you get the following.

① $x + y = 7$

② $3x - 1 = y$

The answer to the puzzle is the pair of numbers that makes both sentences true. You can solve the puzzle by graphing. Look at the diagram and answer the questions below.



A is the graph of $x + y = 7$. Do the coordinates of every point on the line make $x + y = 7$ true? B is the graph of $3x - y = 1$. What can you say about the coordinates of every point on B?

How many points do the two lines have in common? Tell the coordinates. Now give the answer to the puzzle.

The two equations used to solve the puzzle together form what is called a **system of equations,** or a **system of sentences.** To solve a system of two equations, you find the ordered pair(s) of numbers (x, y) that make both equations true.

Solving a system of equations gives the same result as solving a compound sentence using the connective *and*. For example, the compound sentence

$$x + y = 7 \text{ and } 3x - 1 = y$$

has the same solution set as the system shown above. The solution set of the system or of the compound sentence can be described using the notation for set intersection.

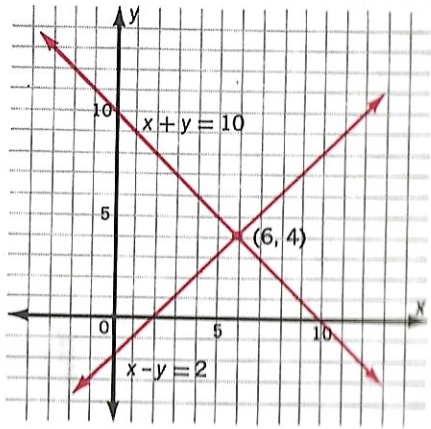
$$\{(x, y) : x + y = 7 \text{ and } 3x - 1 = y\}$$
$$= \{(x, y) : x + y = 7\} \cap \{(x, y) : 3x - 1 = y\}$$

The solution set can be found by graphing, as you saw on page 220.

EXAMPLE. Solve the following system graphically.

$$\begin{cases} x + y = 10 \\ x - y = 2 \end{cases}$$

The two lines on the graph show all points that make either equation true. The point that makes *both* true is shown as a red dot. Its coordinates are (6, 4). To show that (6, 4) makes each equation true, substitute the values for x and y in both equations. Note that 6 is the replacement for x in (x, y) and that 4 is the replacement for y.



Check $6 + 4 = 10$ True.
$6 - 4 = 2$ True.

Solution set: $\{(6, 4)\}$

The two lines are the graphs of

$$\{(x, y) : x + y = 10 \text{ or } x - y = 2\}.$$

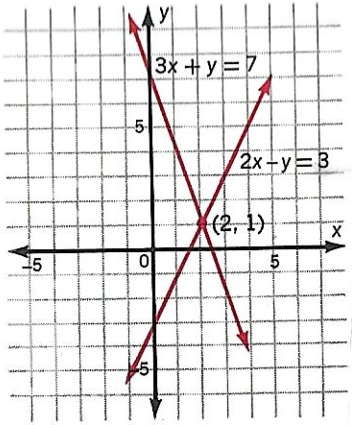
The point of intersection is the graph of

$$\{(x, y) : x + y = 10 \text{ and } x - y = 2\}.$$

Explain.

Now try these

At the right is the graph of the system $\begin{cases} 2x - y = 3 \\ 3x + y = 7 \end{cases}$. Use the graph for Exercises 1–3.



1. $\{(x, y) : 3x + y = 7\} \cap \{(x, y) : 2x - y = 3\} = \{(\underline{?}, \underline{?})\}$

2. Give one ordered pair that makes each of the following sentences false.

a. $2x - y = 3$ *and* $3x + y = 7$

b. $2x - y = 3$ *or* $3x + y = 7$

3. What is the solution set of the system $\begin{cases} 2x - y = 3 \\ 3x + y = 7 \end{cases}$?

Answers: **1.** 2, 1 **2. a.** Any ordered pair except (2, 1). **b.** Any coordinates of points not on the lines. **3.** {(2, 1)}

Checkpoint

1. What is a system of sentences? How would you describe the solution set of a system of sentences?

2. What compound sentence has the same solution set as the system $\begin{cases} 2x + 3y = 5 \\ 4x + 7y = 10 \end{cases}$? as the system $\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$?

3. Is the solution set of a system the intersection or the union of the solution sets of the sentences in the system?

4. How do you solve a system of sentences graphically?

Exercises

A — Draw a graph to find the solution set of each system or compound sentence.

1. $\begin{cases} x + y = 4 \\ 2x - y = 8 \end{cases}$

2. $\begin{cases} 4x - y = 1 \\ y = 2x - 3 \end{cases}$

3. $y = x - 1$ *and* $3x + y = 11$

4. $3x - y = 14$ *and* $2x + y = 6$

5. $x - y = 5$ *and* $x + 5y = 11$

— Use your results from Exercises 1–5 to complete the following.

6. $\{(x, y) : x + y = 4\} \cap \{(x, y) : 2x - y = 8\} = \{(\underline{?}, \underline{?})\}$

7. $\{(x, y) : 4x - y = 1\} \cap \{(x, y) : y = 2x - 3\} = \{(\underline{?}, \underline{?})\}$

8. $\{(x, y) : y = x - 1\} \cap \{(x, y) : 3x + y = 11\} = \{(\underline{?}, \underline{?})\}$

9. $\{(x, y) : 3x - y = 14\} \cap \{(x, y) : 2x + y = 6\} = \{(\underline{?}, \underline{?})\}$

10. $\{(x, y) : x - y = 5\} \cap \{(x, y) : x + 5y = 11\} = \{(\underline{?}, \underline{?})\}$

B — Graph the following sets of points. If the graph is a finite set of points, give the coordinates of each point.

11. $\{(x, y) : x + y = 12\} \cup \{(x, y) : x - y = 3\}$

12. $\{(x, y) : x + y = 12\} \cap \{(x, y) : x - y = 3\}$

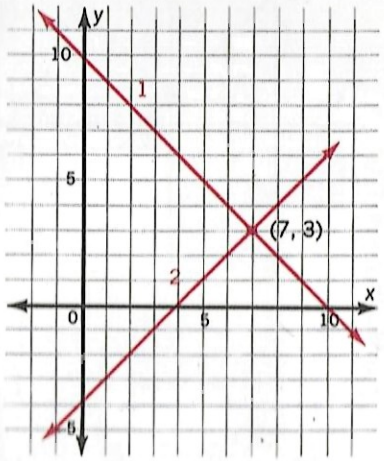
13. $\{(x, y) : x + y < 12\} \cap \{(x, y) : x - y < 3\}$

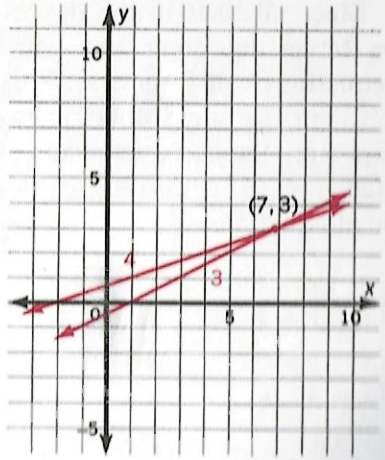
14. $\{(x, y) : x + y > 12\} \cap \{(x, y) : x - y > 3\}$

5–2 Equivalent Systems

Two systems of equations are graphed below. Study the systems and their graphs.

$\begin{cases} x + y = 10 & 1 \\ x - y = 4 & 2 \end{cases}$ $\begin{cases} x - 2y = 1 & 3 \\ x - 3y = -2 & 4 \end{cases}$





What is the solution set of each system? Because both systems have the same solution set, {(7, 3)}, the two systems are **equivalent systems** of equations.

Definition Two systems of equations or inequalities are equivalent if and only if they have the same solution set.

How is the use of the word *equivalent* in *equivalent systems* like its use in *equivalent sentences*? in *equivalent expressions*?

EXAMPLE 1. Show that the following two systems are equivalent.

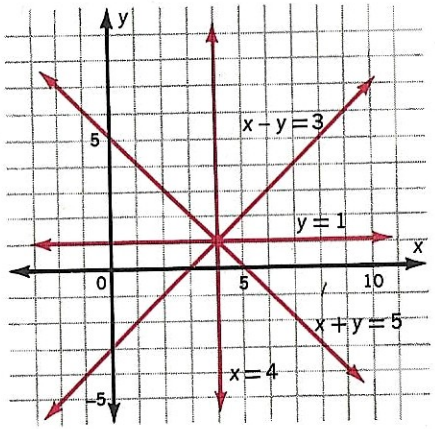
$$\begin{cases} x = 2 & 1 \\ y = 5 & 2 \end{cases} \qquad \begin{cases} 2x + y = 9 & 3 \\ 3x - 2y = -4 & 4 \end{cases}$$

You can find the solution set of the first system by inspection It is $\{(2, 5)\}$. Now check in the second system.

$$\begin{cases} 2(2) + 5 = 4 + 5 = 9 \\ 3(2) - 2(5) = 6 - 10 = -4 \end{cases}$$

Since the two systems have the same solution set, they are equivalent. How can you use a graph to make sure that the solution set of the second system does not contain other points besides (2, 5)?

EXAMPLE 2. Write all equivalent systems of equations using the graphs of four equations at the right.



Since all lines in the figure intersect in one point, any pair of equations will form a system equivalent to any other pair. There are, then, six equivalent systems, each consisting of a pair of equations.

$$\begin{cases} x - y = 3 \\ x + y = 5 \end{cases} \qquad \begin{cases} x - y = 3 \\ y = 1 \end{cases}$$

$$\begin{cases} x - y = 3 \\ x = 4 \end{cases} \qquad \begin{cases} x + y = 5 \\ x = 4 \end{cases}$$

$$\begin{cases} y = 1 \\ x + y = 5 \end{cases} \qquad \begin{cases} y = 1 \\ x = 4 \end{cases}$$

Which system is easiest to solve by inspection?
Show that $\{(4, 1)\}$ is the solution set of each system.

Checkpoint

1. What are equivalent systems of sentences?
2. Tell what kind of system is easiest to solve by inspection.

Exercises

A **1.** Graph the set $\{(x, y) : x = 4\} \cup \{(x, y) : y = 3\}$. What is the solution set of the system $\begin{cases} x = 4 \\ y = 3 \end{cases}$?

2. What ordered pairs of numbers make $x = -2$ *and* $y = 3$ true?

3. What is the solution set of the system $\begin{cases} x = -3 \\ y = 8\frac{1}{4} \end{cases}$?

Find the solution set $\{(x, y)\}$ for each compound sentence.

4. $x = 9$ *and* $y = 2$

5. $x = -2$ *and* $y = -4$

6. $x = -2\frac{1}{6}$ *and* $y = 0$

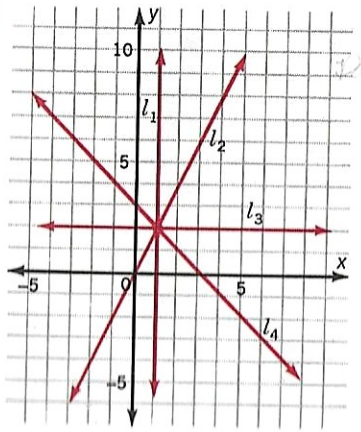
7. $x = -\frac{32}{5}$ *and* $y = \frac{12}{5}$

8. $x = 0$ *and* $y = -6$

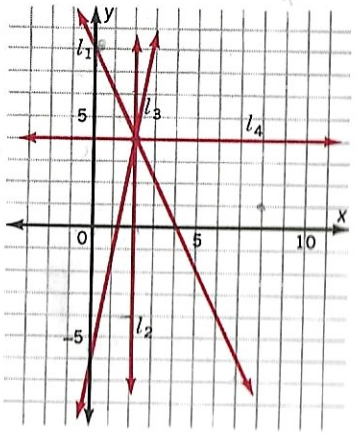
9. $x = -8\frac{1}{6}$ *and* $y = 0$

Four intersecting lines are shown in the figures below each exercise. Which lines are the graphs of the system shown?

10. $\begin{cases} x + y = 3 \\ 2x - y = 0 \end{cases}$



11. $\begin{cases} 2x + y = 8 \\ 5x - y = 6 \end{cases}$



12. Write the system of equations whose graph is l_1 and l_2 in the figure for Exercise 10. What is its solution set?

13. In the graph for Exercise 11, what are the coordinates of the point in $l_3 \cap l_4$? Write the equations for the system of lines $\begin{cases} l_1 \\ l_2 \end{cases}$.

— Show that the system labeled **a** is equivalent to the system labeled **b**.

14. a. $\begin{cases} x + y = 5 \\ x - y = 1 \end{cases}$ **b.** $\begin{cases} x = 3 \\ y = 2 \end{cases}$

15. a. $\begin{cases} 2x + y = 8 \\ x - 2y = 9 \end{cases}$ **b.** $\begin{cases} x = 5 \\ y = -2 \end{cases}$

16. a. $\begin{cases} 2x - 3y = -14 \\ 4x + y = 0 \end{cases}$ **b.** $\begin{cases} x = -1 \\ y = 4 \end{cases}$

17. a. $\begin{cases} x + y = 3 \\ 2x - y = 4 \end{cases}$ **b.** $\begin{cases} x = \frac{7}{3} \\ y = \frac{2}{3} \end{cases}$

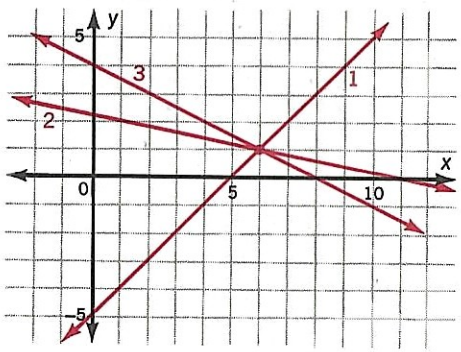
18. a. $\begin{cases} 2x - y = 3 \\ 2x + y = 9 \end{cases}$ **b.** $\begin{cases} 2x = 6 \\ y = 3 \end{cases}$

19. a. $\begin{cases} 2x - y = 4 \\ x + y = 6 \end{cases}$ **b.** $\begin{cases} 3x = 10 \\ 3y = 8 \end{cases}$

B — For Exercises 20 and 21, use the graph at the right to find the solution set of the system labeled **a**. Does the ordered pair in the solution set satisfy the equation labeled **b**? Write two other systems equivalent to **a**.

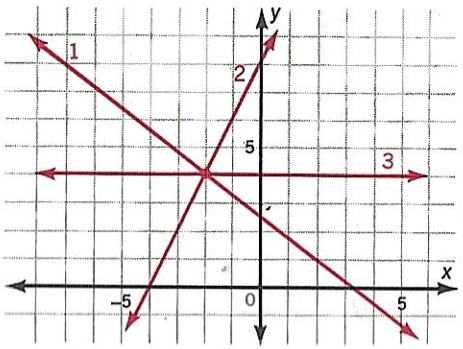
20. a. $\begin{cases} x - y = 5 \\ x + 5y = 11 \end{cases}$ 1
2

b. $2x + 4y = 16$ 3



21. a. $\begin{cases} 6x + 8y = 20 \\ -6x + 3y = 24 \end{cases}$ 1
2

b. $11y = 44$ 3



22. Study Exercises 20 and 21. When you add the left sides of equations 1 and 2 and the right sides of equations 1 and 2, do you get an equation, 3, that is satisfied by the ordered pairs in the solution set of the system?

23. Graph the equation $x + y = 4$. What is the graph of $2x + 2y = 8$? of $5x + 5y = 20$? Is $\{(x, y) : x + y = 4\} \cap \{(x, y) : 2x + 2y = 8\}$ a set with no ordered pairs, one ordered pair, or an infinite number of ordered pairs? Explain.

5–3 Solving Systems of Equations by Addition

Graphic methods for solving systems help you understand the important meaning of the solution set of a system. Algebraic methods are needed as well, because a solution set such as $\{(2\frac{1}{3}, 3\frac{3}{5})\}$ is difficult to read from a graph, and one such as $\{(2\frac{1}{3}, \sqrt{3})\}$ cannot be read exactly at all. Also, algebraic methods are needed to program computers to solve systems.

The first algebraic method is called the addition method.

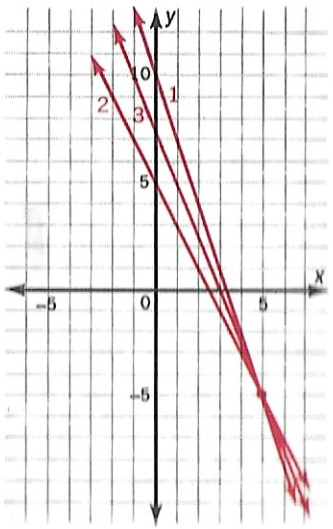
Below are three systems of equations. Is $\{(5, -5)\}$ the solution set of each system? Use substitution to find out.

$\begin{cases} 3x + y = 10 \\ 2x + y = 5 \end{cases}$ $\begin{cases} 3x - y = 20 \\ x - 2y = 15 \end{cases}$ $\begin{cases} x + y = 0 \\ x - y = 10 \end{cases}$

Now take the first pair of equations and add to get a third equation.

$$\begin{array}{rrl} 3x + & y = 10 & 1 \\ +\,2x + & y = 5 & 2 \\ \hline 5x + & 2y = 15 & 3 \end{array}$$

Does $(5, -5)$ satisfy equation 3? Since $(5, -5)$ does satisfy 3, the graph of the new equation passes through the point of intersection of the graphs of the original equations. Any ordered pair (x, y) that makes both 1 and 2 true will make 3 true, and the graphs of the three equations have a point in common.



In general, if you add two equations whose graphs intersect, you will get another equation whose graph passes through the point of intersection.

In solving systems of equations by the addition method, you get an equation whose graph is a line parallel to one of the axes. This is done by eliminating one variable when you add.

EXAMPLE 1. Solve the following system. $\begin{cases} 3x + y = 10 & 1 \\ 2x + y = 5 & 2 \end{cases}$

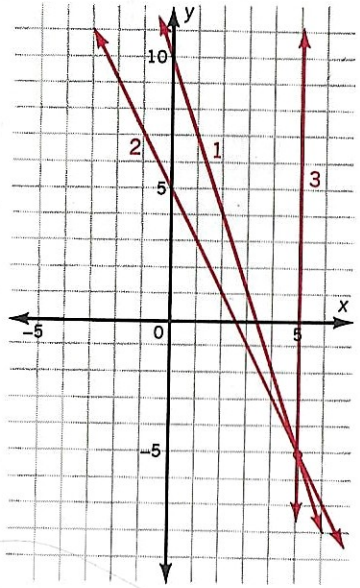
Multiplying both sides of equation 2 by -1 and adding eliminates y and gives you an equation containing only x.

Multiply equation 2 by -1.
Add.

$$\begin{cases} 3x + y = 10 \\ -2x - y = -5 \\ \hline x = 5 \quad 3 \end{cases}$$

Now use equation 1 or 2 and equation 3 as a new, but equivalent, system.

$$\begin{cases} 3x + y = 10 & 1 \\ x = 5 & 3 \end{cases}$$



The first coordinate for every point on the graph of 3 is 5. Hence, the value for x where 1 and 3 intersect is 5. Substitute 5 into 1 to find y.

$$3(5) + y = 10$$
$$y = -5$$

The solution set of $\begin{cases} x = 5 \\ y = -5 \end{cases}$ is $\{(5, -5)\}$. Check to see that the solution set of the original system is also $\{(5, -5)\}$.

EXAMPLE 2. Solve the following system of equations.

$$\begin{cases} 2x + 3y = -1 & 1 \\ 5x - 2y = -12 & 2 \end{cases}$$

To eliminate one variable by addition, the sum of the coefficients of that variable in the two equations must be 0. For this to happen, the coefficients must be negatives of each other. The coefficients of y become negatives of each other if you multiply the sides of equation 1 by 2 and the sides of equation 2 by 3.

Multiply equation 1 by 2.
Multiply equation 2 by 3.
Add.

$$\begin{cases} 4x + 6y = -2 \\ 15x - 6y = -36 \\ \hline 19x = -38 \\ x = -2 \quad 3 \end{cases}$$

Now solve either equivalent system. (Both are done for you here.)

$$\begin{cases} 2x + 3y = -1 & 1 \\ x = -2 & 3 \end{cases} \qquad \begin{cases} 5x - 2y = -12 & 2 \\ x = -2 & 3 \end{cases}$$

Substitute -2 for x in either equation 1 or equation 2.

$$2(-2) + 3y = -1 \qquad 5(-2) - 2y = -12$$
$$y = 1 \qquad y = 1$$

The solution set is $\{(-2, 1)\}$. Check by substituting -2 for x and 1 for y in both equation 1 and equation 2.

Here is a summary of the steps used in the addition method of solving a system of two linear equations in two variables.

1. Make the coefficients of either x or y in both equations negatives of each other by using the multiplication postulate for equations.
2. Add the sides to eliminate x or y, getting a third equation in y or x, respectively.
3. Choose the new equation and one of the original equations as an equivalent system.
4. Substitute the number for x or y in either original equation and solve for the other variable.
5. Check by substituting the number pair in both original equations.

Now try these

In each of the following, what have both sides of the equation in **a** been multiplied by to get the equation in **b**?

	a	b
1.	$3x - 2y = 6$	$6x - 4y = 12$
2.	$-2x + y = -4$	$+8x - 4y = +16$
3.	$3x - 2y = 5$	$6x - 4y = 10$
4.	$5x - 7y = -2$	$15x - 21y = -6$

— Add the two expressions.

5. $3x - 5y$
   $6x + 5y$

6. $4x - 3y$
   $-4x - 6y$

7. $9x - 8y - 5$
   $7x + 8y - 3$

— What have both sides of each equation in **a** been multiplied by to get each respective equation in **b**?

	a	b
8.	$3x - 2y = -4$ $-5x + 3y = -1$	$15x - 10y = -20$ $-15x + 9y = -3$
9.	$3x - 2y = -9$ $3x - 5y = -10$	$15x - 10y = -45$ $-6x + 10y = 20$

Answers: **1.** 2 **2.** −4 **3.** 2 **4.** 3 **5.** $9x$ **6.** $-9y$ **7.** $16x - 8$ **8.** 5; 3 **9.** 5; −2

Exercises

A — Solve the following systems of equations by addition.

1. $\begin{cases} 3x + y = 9 \\ 2x - y = 1 \end{cases}$
2. $\begin{cases} 3x + y = 10 \\ 2x + y = 7 \end{cases}$
3. $\begin{cases} 5r - s = -23 \\ 3r - s = -15 \end{cases}$
4. $\begin{cases} 2a + 3b = -5 \\ 5a + 3b = 1 \end{cases}$
5. $\begin{cases} 3x + 2y = -7 \\ 5x - 2y = -1 \end{cases}$
6. $\begin{cases} x + 3y = 14 \\ x - 2y = -1 \end{cases}$
7. $\begin{cases} 2m + 3n = 6 \\ 2m - 5n = 22 \end{cases}$
8. $\begin{cases} 2x + 3y = 8 \\ 3x + y = 5 \end{cases}$
9. $\begin{cases} 2x + y = 3 \\ 7x - 4y = 18 \end{cases}$
10. $\begin{cases} -4x - y = -16 \\ 5x - 3y = 3 \end{cases}$

— Solve the following compound sentences.

11. $5x + y = 15$ *and* $3x + 2y = 9$
12. $7x - 5y = -2$ *and* $-8x - y = 9$
13. $4p - 2q = 20$ *and* $p + 5q = -17$
14. $2x + 5y = 18$ *and* $-5x + y = -18$
15. $2x + 2y = 8$ *and* $5x - 3y = 4$
16. $4a + 3b = -2$ *and* $8a - 2b = 12$
17. $2x - 5y = 7$ *and* $3x - 2y = -17$
18. $9c + 7d = 14$ *and* $-6c - d = -2$
19. $4x + 3y = 7$ *and* $4x + 4y = 12$
20. $5a + 9b = 6$ *and* $6a + 5b = 13$

— List the members for each of the following sets.

21. $\{(x, y) : 2x + y = 8\} \cap \{(x, y) : 4x - y = -8\}$
22. $\{(x, y) : 2x + 3y = 9\} \cap \{(x, y) : 4x + 3y = 21\}$
23. $\{(a, b) : 4a + 3b = 0\} \cap \{(a, b) : -2a - b = 2\}$
24. $\{(x, y) : 5x + 3y = 2\} \cap \{(x, y) : 3x + 5y = -2\}$
25. $\{(m, n) : 5m + 9n = 61\} \cap \{(m, n) : 7m + 3n = 47\}$
26. $\{(x, y) : 7x - 2y = 55\} \cap \{(x, y) : -9x + 5y = -78\}$
27. $\{(a, b) : 2a + 3b = 0\} \cap \{(a, b) : 2b - 3a = 26\}$
28. $\{(x, y) : 5x - 6y = 1\} \cap \{(x, y) : 3x = 2y - 33\}$
29. $\{(r, s) : 11r = 3s + 2\} \cap \{(r, s) : 7r + 70 = 7s\}$
30. $\{(x, y) : 4x - 5y = 2\} \cap \{(x, y) : -8x - 3y = -30\}$

B — Solve the following systems of equations by addition.

31. $\begin{cases} 9x - 2y = 2\frac{1}{2} \\ 5x - 6y = -3\frac{1}{2} \end{cases}$
32. $\begin{cases} 3x + 5y = 4\frac{1}{2} \\ -9x - 2y = -7 \end{cases}$
33. $\begin{cases} 8x + 6y = 10 \\ -4x + 3y = -1 \end{cases}$
34. $\begin{cases} 5x + 4y = -\frac{1}{3} \\ 7x + 2y = \frac{4}{3} \end{cases}$
35. $\begin{cases} 2x + 3y = 12.6 \\ -5x + 9y = -12.4 \end{cases}$
36. $\begin{cases} x + 3y = 17.3 \\ 2x - 7y = -36.9 \end{cases}$
37. $\begin{cases} -6x - 5y = -4.6 \\ 10x + 3y = 6.6 \end{cases}$
38. $\begin{cases} 2x + 7y = 52 \\ -3x + 5y = -16 \end{cases}$

— Solve the following compound sentences.

39. $2x - 7y = 8$ *and* $4y - 9x = 19$
40. $4x - 6y = 8$ *and* $-9x - 6y = -96$
41. $2x - 3y = 8$ *and* $3x - 7y = 7$
42. $-2x - y = -7$ *and* $2x - y = 5$
43. $-3x - y = -10\frac{1}{2}$ *and* $6x - 2y = 7$
44. $2x - 3y = -14$ *and* $3x + 7y = 48$
45. $ax + 3y = 8a$ *and* $ax - y = 4a$
46. $7x + ay = 13a$ *and* $2x - ay = 5a$

C — Solve the following systems of equations by addition.

47. $\begin{cases} 2x + 1.5y = -1 \\ 0.5x + 2y = 3 \end{cases}$
48. $\begin{cases} x - 2.3 = -5y \\ 2x - 2 = 3y \end{cases}$
49. $\begin{cases} 2a - 1.5b = 10 \\ 0.3a - 0.05b = 0.8 \end{cases}$
50. $\begin{cases} 2x - 3y = 0.1 \\ 3x - 3 = -5y \end{cases}$
51. $\begin{cases} 1.25s + 8.25t = 107.5 \\ 2.5s - t = 0.8 \end{cases}$
52. $\begin{cases} 10x + 5y = 3.5 \\ 3y - 0.6 = 3x \end{cases}$

53. $\begin{cases} 2.75m + n = 3.275 \\ 1.25m + 0.5n = 1.625 \end{cases}$ **54.** $\begin{cases} 3s - 0.7t = 26.3 \\ 0.7s + t = 7.3 \end{cases}$

— List the members of each of the following sets.

55. $\{(x, y) : x - 1.6y = 0.38\} \cap \{(x, y) : 0.4x + 3y = 0.88\}$

56. $\{(a, b) : a - 2.33b = 0.67\} \cap \{(a, b) : a + 0.2b = 3.2\}$

57. $\{(x, y) : 2x - 0.2y = 4\} \cap \{(x, y) : 3x + y = 11.2\}$

58. $\{(a, b) : 6a + 0.3b = 1.2\} \cap \{(a, b) : 0.1 + 3a = 0.2b\}$

59. $\{(c, d) : 1.69d - 12 = 7c\} \cap \{(c, d) : 3c + 6d = 62.1\}$

60. $\{(x, y) : 1.7x + 1.3y = 6.8\} \cap \{(x, y) : 0.2x + 0.3y = -2.4\}$

5–4 Using Systems of Equations

Many verbal problems are easily solved using systems of equations. The procedure is similar to the method in Chapter 3, but now each condition is expressed with a single equation in two variables.

EXAMPLE 1. **The difference between two numbers is 6.** *Twice the larger is 5 more than 3 times the smaller.* Find the numbers.

① Let the larger number be x and the smaller number be y.

② Express each condition in an equation.

The difference is 6.	$x - y = 6$	1
Twice the larger is 5 more than 3 times the smaller.	$2x = 3y + 5$	2

③ Solve the systems of equations.

Multiply 1 by -3.	$-3x + 3y = -18$
Rewrite 2.	$2x - 3y = 5$
Add.	$-x = -13$
	$x = 13$ 3

Solve either the system of equations 1 and 3 or of 2 and 3.

$$13 - y = 6$$
$$-y = -7$$
$$y = 7$$

④ Check to see that 13 and 7 satisfy the two conditions of the problem.

$13 - 7 = 6$ $\qquad$ $2(13) = 3(7) + 5$, or 26

EXAMPLE 2. At a high school play, **90 adult and student tickets were sold.** How many of each could have been sold?

① Let the number of student tickets be x and the number of adult tickets be y.

② Use the condition given to write an equation.

90 adult and student tickets were sold. $x + y = 90$

③ Since only one condition is given in the problem, the solution set is as follows.

$$\{(0, 90), (1, 89), (2, 88), \cdots, (88, 2), (89, 1), (90, 0)\}$$

The problem limits the domain to whole numbers. Why?

EXAMPLE 3. Suppose that the condition in Example 2 is true, and suppose further that you know that *student tickets cost 50 cents and adult tickets cost 80 cents, and the total receipts were $48.00.* How many of each kind were sold?

① Use the new condition to write an equation in x and y, where x and y have the same meaning as in Example 2.

The number of cents from the sale of x student tickets is $50x$.
The number of cents from the sale of y adult tickets is $80y$.
The total receipts are $48.00, or 4800 cents.

$$50x + 80y = 4800 \quad 2$$

② Combine this with the equation $x + y = 90$ to form a system.

$$\begin{cases} 50x + 80y = 4800 & 2 \\ x + y = 90 & 1 \end{cases}$$

③ Complete the solution of the system.

④ Check the solution set with the conditions of the problem. Do you find $\{(80, 10)\}$ to be the solution set?

Exercises

A — Write two equations for each of the problems below and solve the resulting system by addition. Check your results.

1. The difference between two numbers is 4. Three times the larger is 2 more than five times the smaller. Find the numbers.

2. The sum of two numbers is 20. Twice one of them is three times the other. Find the numbers.

3. The width of a rectangle is 6 inches less than the length. The perimeter is 72 inches. Find the length and the width.

4. Find two numbers such that four times the first number minus five times the second number equals 100, and twice the first number plus the second number equals 8.

5. At a high school play, students paid 35 cents and adults paid 70 cents for admission. The total receipts for 100 tickets amounted to $48.30. How many of each kind were sold?

6. Six apples and one pear cost 37 cents. At the same price three apples and six pears cost 57 cents. Find the cost of each.

7. Nancy is paid $4.00 a week more than Helen. In 11 weeks Helen earns as much as Nancy does in 10 weeks. What is the weekly wage of each?

8. The sum of two numbers is 16. Twice their difference increased by 3 is 7. Find the numbers.

9. Five full-fare tickets and one half-fare ticket on the railroad cost $90.75, and four full-fare and two half-fare tickets cost $82.50. How much does a full-fare ticket cost? a half-fare ticket?

10. Phil's father is 25 years older than Phil. In 10 years he will be twice as old as Phil. What are their present ages? (*Hint:* Write one equation for their present ages and another for their ages 10 years from now.)

B **11.** Edith's present age is 2 less than twice her sister Jane's. In 5 years Edith's age will equal three times Jane's age now. What are their present ages?

12. Seven times the sum of two numbers is 28. If the second number is multiplied by 3 and added to the first number, the sum equals the second number. What are the two numbers?

13. The length of a rectangle is 3 inches more than twice the width. The perimeter is 45 inches. Find the length and the width.

C **14.** Find two numbers such that one is as much more than 30 as the other is less than 30, and eight times their difference is equal to the larger.

The Problem of the Twelve Coins

Of 12 coins, one is counterfeit and is either lighter or heavier than the other 11. Determine in three weighings on a platform balance which is the counterfeit coin, and tell if it is heavier or lighter than the other 11.

5–5 Solving Systems of Equations by Substitution

If the system at the right has an ordered pair of numbers in its solution set, call this pair (x, y). You can solve for x or for y in 1 and substitute in 2 to find the replacement for (x, y) that makes both equations true. This is solving systems of equations by **substitution.**

$$\begin{cases} 4x + y = 2 & 1 \\ 2x + 3y = -4 & 2 \end{cases}$$

① Solve for x or for y in either equation in terms of the other variable. Solve equation 1 for y.

$$\begin{aligned} 4x + y &= 2 \\ y &= 2 - 4x \end{aligned}$$

② Replace the variable y in the other equation with the expression $2 - 4x$. This gives an equation in x.

$$\begin{aligned} 2x + 3y &= -4 \\ 2x + 3(2 - 4x) &= -4 \\ 2x + 6 - 12x &= -4 \\ -10x &= -10 \\ x &= 1 \qquad 3 \end{aligned}$$

③ Solve the system consisting of 1 and 3 or that consisting of 2 and 3

$$\begin{aligned} 4(1) + y &= 2 \\ 4 + y &= 2 \\ y &= -2 \end{aligned} \qquad \begin{aligned} 2(1) + 3y &= -4 \\ 3y &= -6 \\ y &= -2 \end{aligned}$$

④ If the numbers obtained for the variables check in the original system, then they form the solution set. Is the solution set $\{(1, -2)\}$?

EXAMPLE. Solve the following system of equations by substitution.

$$\begin{cases} -x + 2y = 4 & 1 \\ 5x - 3y = 1 & 2 \end{cases}$$

① Solve equation 1 for x in terms of y.

$$\begin{aligned} -x &= -2y + 4 \\ x &= 2y - 4 \end{aligned}$$

② Replace x in equation 2 with $2y - 4$ to find an equation for y.

$$\begin{aligned} 5x - 3y &= 1 \\ 5(2y - 4) - 3y &= 1 \\ 10y - 20 - 3y &= 1 \\ 7y &= 21 \\ y &= 3 \qquad 3 \end{aligned}$$

③ Use equation 3 with 1 or 2.

$$\begin{aligned} -x + 2(3) &= 4 \\ -x + 6 &= 4 \\ x &= 2 \end{aligned} \qquad \begin{aligned} 5x - 3(3) &= 1 \\ 5x - 9 &= 1 \\ x &= 2 \end{aligned}$$

④ Check to see if $\{(2, 3)\}$ is the solution set of the system.

A — Solve the following systems of equations by substitution.

1. $\begin{cases} x = y + 4 \\ 2x - 5y = 2 \end{cases}$
2. $\begin{cases} y = 3 - x \\ 5x + 3y = -1 \end{cases}$
3. $\begin{cases} x = 8y \\ x - 4y = 12 \end{cases}$
4. $\begin{cases} 5x - y = 1 \\ 3x + 2y = 13 \end{cases}$
5. $\begin{cases} 5a = b \\ a = 10b + 5 \end{cases}$
6. $\begin{cases} r = 5 - s \\ 2r + 7s = 0 \end{cases}$
7. $\begin{cases} c = d + 4 \\ 2c + 3d = -2 \end{cases}$
8. $\begin{cases} x + 3y = 5 \\ -3x + 2y = 18 \end{cases}$
9. $\begin{cases} c = d - 4 \\ 3c + d = 4 \end{cases}$
10. $\begin{cases} y = \frac{1}{4}x \\ 3x + 4y = 32 \end{cases}$

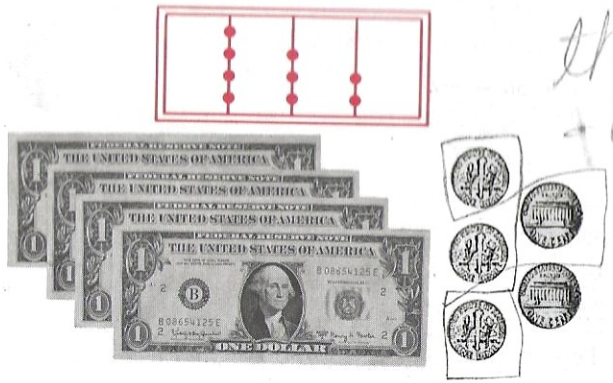
— Use substitution to find the solution sets of the following compound sentences.

11. $3x + y = 7$ *and* $x - y = 1$
12. $4x - y = -7$ *and* $5x - 8y = -2$
13. $x + y = 3$ *and* $x - y = 1$
14. $2x + y = 5$ *and* $8x - y = 45$
15. $x = 3y$ *and* $2x - 5y = 4$
16. $5x - 3y = -1$ *and* $-x + 3y = -7$
17. $y = -2x$ *and* $x + 2y = 9$
18. $3x - y = -24$ *and* $7x - 9y = 14$

B — List the elements for each set.

19. $\{(x, y) : y = 8 - x\} \cap \{(x, y) : 4x - 3y = -3\}$
20. $\{(x, y) : 3x + 2y = 16\} \cap \{(x, y) : 7x + y = 19\}$
21. $\{(x, y) : y + 2x = 41\} \cap \{(x, y) : 5y - 3x = 14\}$
22. $\{(x, y) : 2x + y = 11\} \cap \{(x, y) : x - y = 2\}$
23. $\{(x, y) : 4x - 2y = 3\} \cap \{(x, y) : 3x - y = 4\}$
24. $\{(x, y) : 5y - x = -9\} \cap \{(x, y) : 4y + 3x = -11\}$
25. $\{(x, y) : x - 5y = -2\} \cap \{(x, y) : 9x + 7y = 34\}$
26. $\{(a, b) : 4a + b = 8\} \cap \{(a, b) : 5a + 3b = 3\}$
27. $\{(u, v) : 2u - 3v = -12\} \cap \{(u, v) : 7u + v = 19\}$
28. $\{(s, t) : 4s + t = 8\} \cap \{(s, t) : s + 4t = 6\}$

In the decimal place-value numeration system, the ten digits are symbols for the whole numbers less than the base. The digits are the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The place in which a digit occurs in a numeral determines the value it represents. The abacus and diagram below represent the important idea of place value in a decimal system by showing the value of 432.



Since the digit 4 is in the second place to the left of the units place, it represents 4 hundreds; similarly, the digit 3 represents 3 tens; and the digit 2 represents 2 units, or 2 ones.

If you were told to write the numeral with 7 as the tens digit and 4 as the units digit, you would write 74. Or if you were told to write the numeral with 4 as the tens digit and 7 as the units digit, you would write 47. A numeral with 4 as the tens digit and 7 as the units digit names 4(10) + 7, or 47.

You can use this idea of place value to solve many interesting problems.

EXAMPLE. The sum of the numbers named by the digits of a two-digit numeral is 15. The number named is 9 less than the number named when the digits are reversed. Find the number.

① Let t be the tens digit and u be the units digit.

② The number is $10t + u$ and the number formed when the digits are reversed is $10u + t$. Using the conditions of the problem gives the following system.

$$\begin{cases} t + u = 15 \\ 10t + u = 10u + t - 9 \end{cases}$$

③ Solve the system for t and u. The number should be 78.

1. If 7 is the tens digit of a number and u is the units digit, which expression names the number?

a. $7(10) + u$ **b.** $u(10) + 7$ **c.** $7u$

2. Write the expression for the number named when t is the tens digit and 4 is the units digit.

3. Write the expression for the number named when the digits in Exercise 2 are reversed.

4. If t is the tens digit of a number and u is the units digit, write the expression for the number.

5. Write the expression for the number formed when the digits in Exercise 4 are reversed.

Answers: **1.** a **2.** $t(10) + 4$, or $10t + 4$ **3.** $4(10) + t$, or $40 + t$ **4.** $t(10) + u$, or $10t + u$ **5.** $u(10) + t$, or $10u + t$

Exercises

A — Write two equations for each problem. Solve the resulting system by addition or substitution, whichever you choose.

1. The sum of the numbers named by the two digits of a numeral is 9. The number named by the numeral is 27 more than the number named when the digits are reversed. Find the number.

2. Find the number named by two digits whose units digit names a number that is twice that named by the tens digit; the difference between the numbers named by the digits is 4.

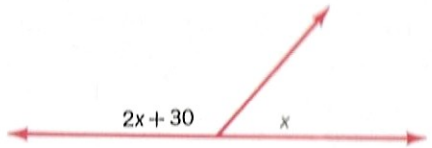
3. The units digit of a two-digit numeral names a number 2 less than that named by the tens digit. The number named by the numeral is 2 more that 6 times the sum of the numbers named by the digits. Find the number.

4. A certain two-digit numeral represents a number that is 5 times the sum of the numbers named by its digits. The digits from left to right name consecutive integers. Find the number.

5. A certain two-digit numeral names a number 5 times the sum of the numbers named by its digits. If 9 were added to the number named by the numeral, the digits in the numeral would be reversed. Find the number.

6. One number exceeds another by 62. If 5 times the smaller is decreased by 2 times the larger, the difference is 155. What are the two numbers?

7. Two angles are supplementary (the sum of their measures is 180°). One of the angles measures 30° greater than 2 times the measure of the other. How many degrees do each of the two angles measure?



8. The perimeter of a rectangular playground is 480 meters. What are the length and width of the rectangle if twice the width is 30 meters more than the length?

B

9. The sum of two amounts of money is \$5000. If one amount is invested at 4% annual interest and the second at 7% annual interest, the total annual interest will be equal to the interest on \$5000 at 6% for one year. What are the amounts invested? (*Hint:* If x and y represent the two amounts, you get the following.

$$\begin{cases} x + y = 5000 \\ 0.04x + 0.07y = 0.06(5000) \end{cases}$$

You can multiply the second equation by 100 and rewrite as

$$\begin{cases} x + y = 5000 \\ 4x + 7y = 30{,}000 \end{cases}$$

for ease in solving.)

10. The Smith Motor Company's total sales on new and used cars this year was \$140,000. The total profit amounted to 10% of the sales. The profit on new cars was 8% and on used cars was 15%. What was the amount taken in on new-car sales? on used-car sales?

11. Mrs. Bunch is considering buying a television set or a stereo phonograph. The combined cost is \$550. Two stereo phonographs cost \$170 more than one television set. How much does each item cost?

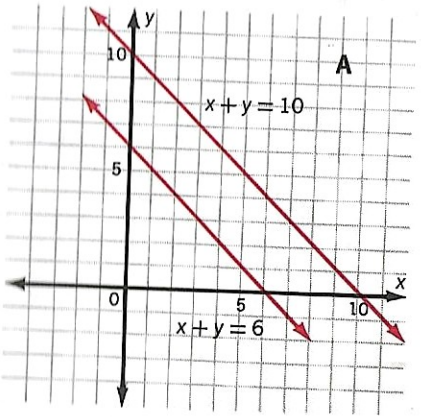
12. Ken Cowing is now 6 years older than his youngest brother. In 8 years, 3 times Ken's age will equal 4 times his brother's age. What are their present ages?

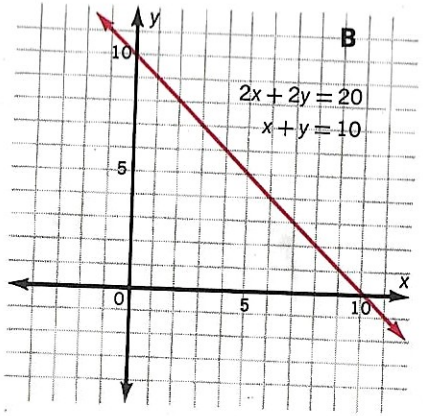
13. Ken Levy's coin collection contains 7 more dimes than nickels. In all, the collection amounts to \$3.25. How many of each coin does he have?

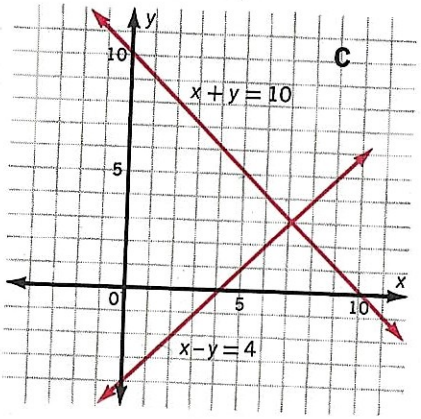
for 80 cents a pound. How many pounds of candy at 60 cents and at $1.20 a pound must be used to make the desired mixture?

15. Tickets for 470 seats in an auditorium were sold for a play. The total sales was $346.50. How many tickets were sold to adults at 90 cents and how many to children at 40 cents?

5–7 Systems of Equations with Many and with No Solutions







Above and at the right are the graphs of three systems of equations. Which has in its solution set one ordered pair? an infinite number of ordered pairs? no ordered pairs?

How do the slopes of the lines compare in each system? How do the y intercepts compare?

Systems of equations are classified by the number of pairs in their solution sets. The table on the next page is based on the graphs shown above. Use the following arbitrary system in considering the relations between constants.

$$\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0 \end{cases}$$

System	A	B	C
Number of solutions	None.	Infinite.	One.
Graphs	Parallel lines: slopes equal, intercepts different.	Same line.	Distinct, intersecting lines.
Relations between constants	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ $\frac{1}{1} = \frac{1}{1} \neq \frac{-10}{-6}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ $\frac{2}{1} = \frac{2}{1} = \frac{-20}{-10}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ $\frac{1}{1} \neq \frac{-1}{1} \neq \frac{10}{4}$
Classification	Inconsistent.	Consistent and dependent.	Consistent and simultaneous, or independent.

Use the statements in the table to try to formulate definitions of *inconsistent*, *consistent*, *dependent*, and *simultaneous*, or *independent*, *systems of equations* before reading the definitions below.

Definition A system of equations or inequalities whose solution set is $\emptyset$ is an inconsistent system.

Definition A system is consistent if it has one or more ordered pairs in the solution set.

Definition A consistent system of equations whose solution set has an infinite number of ordered pairs is a dependent system.

Definition A consistent system of equations with one and only one ordered pair in the solution set is a simultaneous, or independent, system.

Note what happens in the examples on the following page when you use algebraic methods with inconsistent systems and with dependent systems.

EXAMPLE 1. Solve the following system by addition.

$$\begin{cases} x + y = 10 & 1 \\ x + y = 6 & 2 \end{cases}$$

If there is a solution, *then* the sum of the left sides and the sum of the right sides give an equation whose graph passes through the point of intersection of the system. Multiply 2 by -1 and add.

$$\begin{cases} x + y = 10 & 1 \\ -x - y = -6 \end{cases}$$
$$0 \cdot x + 0 \cdot y = 4$$
$$0 = 4 \quad 3$$

No pair will satisfy either of the following systems.

$$\begin{cases} x + y = 10 & 1 \\ 0 = 4 & 3 \end{cases} \qquad \begin{cases} x + y = 6 & 2 \\ 0 = 4 & 3 \end{cases}$$

Equation 3 is not true for any ordered pair (x, y). Hence, the system 1 *and* 2 is inconsistent, and the following is true.

$$\{(x, y) : x + y = 10\} \cap \{(x, y) : x + y = 6\} = \emptyset$$

When you use algebraic methods with a system and get an equation that is true for no value of x, the system is inconsistent.

EXAMPLE 2. Solve the following system by addition.

$$\begin{cases} 2x - 3y = 10 & 1 \\ 10x - 15y = 50 & 2 \end{cases}$$

Multiply equation 1 by -5.
$$\begin{cases} -10x + 15y = -50 \\ 10x - 15y = 50 & 2 \end{cases}$$

Add.
$$0 \cdot x + 0 \cdot y = 0$$
$$0 = 0 \quad 3$$

If 1 *and* 2 has a solution, then it is the same as the solution of 1 *and* 3 or of 2 *and* 3.

$$\begin{cases} 2x - 3y = 10 & 1 \\ 0 = 0 & 3 \end{cases} \qquad \begin{cases} 10x - 15y = 50 & 2 \\ 0 = 0 & 3 \end{cases}$$

There is an infinite number of pairs that will satisfy either system. There is also an infinite number of pairs that will satisfy the original system.

When you use algebraic methods with a system and get an equation that is an identity, the system is dependent.

Checkpoint

1. How many ordered pairs are in the solution set of an inconsistent system? a dependent system?
2. Describe the graph of an inconsistent system; a dependent system.
3. What happens when you use algebraic methods to find the solution set of an inconsistent system? a dependent system? a simultaneous, or independent, system?

Exercises

A — Find the slope and the y intercept of each equation below. Then use your results and write *Independent*, *Inconsistent*, or *Dependent* to classify each system.

1. $\begin{cases} 3x + y = 7 \\ 3x - y = 5 \end{cases}$
2. $\begin{cases} 3x + y = 7 \\ 3x + y = 5 \end{cases}$
3. $\begin{cases} 3x + y = 7 \\ 6x + 2y = 14 \end{cases}$
4. $\begin{cases} 3x + y = 7 \\ 6x + 2y = 10 \end{cases}$
5. $\begin{cases} x + y = 10 \\ x - y = 4 \end{cases}$
6. $\begin{cases} x + y = 10 \\ 4x + 4y = 4 \end{cases}$
7. $\begin{cases} 2x + y = 7 \\ 2x - y = 7 \end{cases}$
8. $\begin{cases} 6x + 8 = 2y \\ y = 3x + 4 \end{cases}$
9. $\begin{cases} x + y = 1 \\ y = -x \end{cases}$
10. $\begin{cases} 4x + 5y = 7 \\ 5x = 7 + 4y \end{cases}$
11. $\begin{cases} 2x - y = 2 \\ 4x - 2y = 0 \end{cases}$
12. $\begin{cases} 3x + 5y = 4 \\ 12 - 9x = 15y \end{cases}$

— Solve the following systems.

13. $\begin{cases} 2x + 3y = 8 \\ x + y = 3 \end{cases}$
14. $\begin{cases} m + n = 14 \\ 28 - 2m = 2n \end{cases}$
15. $\begin{cases} x + y = 9 \\ x - y = 9 \end{cases}$
16. $\begin{cases} y = 2x - 8 \\ 2x - y = 10 \end{cases}$
17. $\begin{cases} y = -4x + 5 \\ 8x + 2y = 10 \end{cases}$
18. $\begin{cases} x = y + 5 \\ 2x - 2y = 7 \end{cases}$

19. $\begin{cases} \frac{2}{3}x - 4 = 3y \\ 6y + 8 = 5x \end{cases}$

20. $\begin{cases} y = 4x - 2 \\ x - 4y = 2 \end{cases}$

21. $\begin{cases} 4a + 3b = 2 \\ 2a + 3b = 4 \end{cases}$

22. $\begin{cases} 5u - 3v = 7 \\ 25u - 15v = 49 \end{cases}$

B — Write *Independent*, *Inconsistent*, or *Dependent* to classify each system.

23. $\begin{cases} ax + by = c \\ ax + by = d, \ c \neq d \end{cases}$

24. $\begin{cases} ax + by = c \\ nax + nby = nc, \ n \neq 0 \end{cases}$

25. $\begin{cases} ax + by = c \\ ax - by = c, \ b \neq 0 \end{cases}$

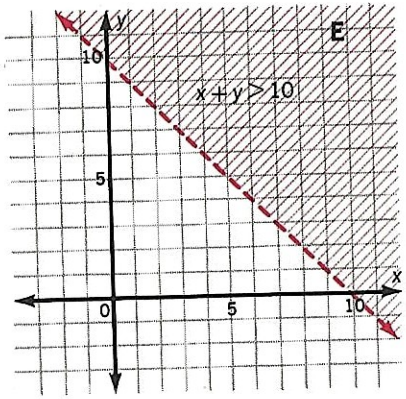
26. $\begin{cases} ax + by = c \\ nax + nby = nd, \ c \neq d, \ n \neq 0 \end{cases}$

5–8 Systems of Inequalities

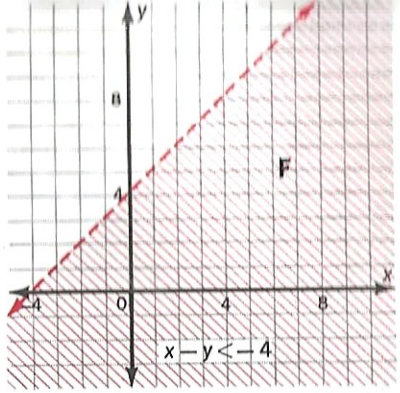
The solution set of the system of inequalities shown at the right is the set of ordered pairs that make both sentences true. Examine the graph of each inequality.

$$\begin{cases} x + y > 10 & 1 \\ x - y < -4 & 2 \end{cases}$$

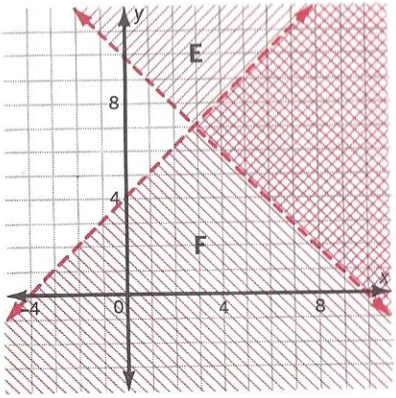
① Any point in half-plane E has coordinates that make $x + y > 10$ true. Note that the graph of $x + y = 10$ is shown dashed to show that it is not included in the half-plane.



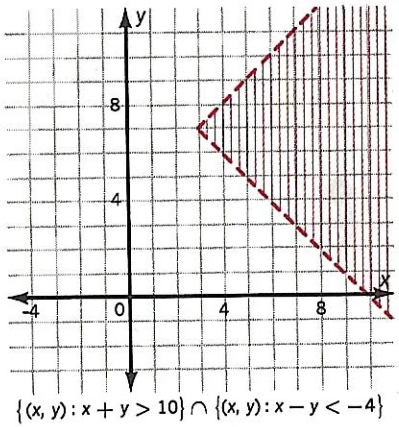
② Any point in half-plane F has coordinates that make $x - y < -4$ true.



③ To show the points that are in both half-planes and whose coordinates are in both solution sets, draw both graphs on the same axes. The points in the intersection are indicated by the crosshatching.



The points with coordinates in the solution set may be graphed as shown in the diagrams. Since there is an infinite number of ordered pairs in the solution set, you could not possibly list them all. One way to indicate the solution set is by a graph.



Exercises

A — Graph each sentence in the system. Shade the graph of the solution set darker than the other points. Write the coordinates of at least three points in the solution set, unless it is empty.

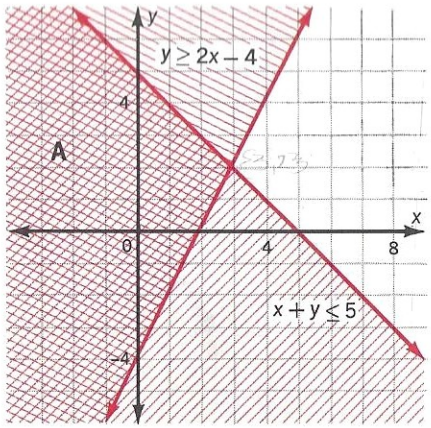
Example. $\begin{cases} y \geq 2x - 4 \\ x + y \leq 5 \end{cases}$

The graph of $y \geq 2x - 4$ is the graph of $y = 2x - 4$ and all points on the left side of that line.

The graph of $x + y \leq 5$ is the graph of $x + y = 5$ and all points on the left side of that line.

The graph of the solution set of the system is the set of points in both inequalities, which is shown by the crosshatching.

Three points in the solution set are (0, 0), (1, 2), and (2, 3).



1. $\begin{cases} x + y \leq 2 \\ x - y \leq 4 \end{cases}$
2. $\begin{cases} y > 3x \\ y \leq -3x \end{cases}$
3. $\begin{cases} x - y = 8 \\ x + y > 3 \end{cases}$
4. $\begin{cases} x + y > 10 \\ x + y < 4 \end{cases}$
5. $\begin{cases} y > 3x \\ y > 3x + 2 \end{cases}$
6. $\begin{cases} 2x + 3y \geq 6 \\ y \leq 4 \end{cases}$

B — For the following compound sentences, shade only the points in the graph of the solution set. Do not shade the graphs of the parts unless these points are also in the graph of the compound sentence.

7. $x \geq 0$ *or* $2x - 3y \leq 6$
8. $x \geq 0$ *and* $2x - 3y \leq 6$
9. $x + y \leq 10$ *or* $x + y \geq -10$
10. $x + y \leq 10$ *and* $x + y \geq -10$

More Challenging Problems

— Is there a pair of numbers that will satisfy each group of three equations? If so, find the number pair.

1. $\begin{cases} x - 5y = 27 \\ 2x + 3y = 2 \\ 5x + 8y = 3 \end{cases}$
2. $\begin{cases} 9x + 4y = 0 \\ 5x - 2y = 6 \\ 3x + 14y = 2 \end{cases}$

— Solve for x and y in terms of a, b, and c.

3. $\begin{cases} ax + by = c \\ bx - ay = 7c \end{cases}$
4. $\begin{cases} ax + by + a^2 + b^2 = 0 \\ bx + ay + 2ab = 0 \end{cases}$
5. $\begin{cases} cy + (b + c)x = 2bc \\ by + (b - c)x = 2^2 - b^2 \end{cases}$

6. Lum is twice as old as Moe was when Lum was Moe's age. When Moe is as old as Lum now is, the sum of their ages will be 100. How old is each?

CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

system of sentences (*221*)
equivalent systems (*223*)
inconsistent system (*241*)
consistent system (*241*)
dependent system (*241*)
simultaneous, or independent, system (*241*)

2. How can you prove that (7, 2) is the solution set of the following system?

$$\begin{cases} x - y = 5 \\ -2x + 3y = -8 \end{cases}$$

3. Which of the following systems are equivalent to the system in Exercise 2? How do you know?

a. $\begin{cases} x = 7 \\ y = 2 \end{cases}$
b. $\begin{cases} x + 2y = 11 \\ x + y = 9 \end{cases}$
c. $\begin{cases} x - y = 5 \\ (x - y) + (2x + 3y) = 5 - 8 \end{cases}$
d. $\begin{cases} 2x + 3y = -8 \\ (x - y) + (2x + 3y) = -3 \end{cases}$
e. $\begin{cases} x - y - 5 = 0 \\ y = 2 \end{cases}$
f. $\begin{cases} x - y = 5 \\ 2x + 3(x - 5) = -8 \end{cases}$

$x - y = 5$ *and* $2x + 3y = -8$

is ?.

5. Complete.

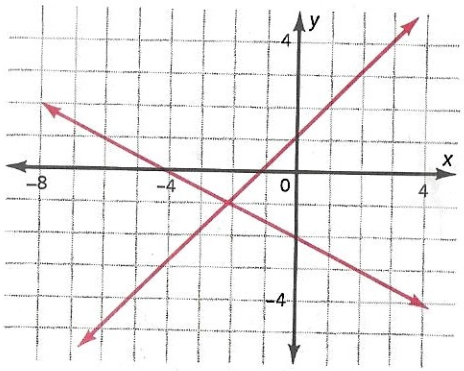
$$\{(x, y) : x - y = 5\} \cap \{(x, y) : 2x + 3y = -8\} = \text{?}$$

6. Is (7, 2) in the solution set of $x - y = 5$ *or* $2x + 3y = -8$?

7. Find two systems equivalent to the following.

$$\begin{cases} x + y = -4 \\ 2x - 5y = 8 \end{cases}$$

8. Shown below is the graph of a system of two equations. What is the solution set of the system?



Tell whether each of the following systems has one and only one, none, or an infinite number of ordered pairs in its solution set.

9. $\begin{cases} y = 2x - 6 \\ 2y = x + 5 \end{cases}$

10. $\begin{cases} 3x - y = 8 \\ 2y = 6x - 16 \end{cases}$

11. $\begin{cases} 2x + 3y = 7 \\ 6y - 13 = 4x \end{cases}$

12. $\begin{cases} 2x - 3y = 5 \\ 4x = 10 - 6y \end{cases}$

13. How would you describe the graphs in Exercises 9–12; that is, for which do you get distinct intersecting lines, for which do you get parallel lines, and for which do you get a single line?

14. Classify the systems in Exercises 9–12 as inconsistent, dependent, or simultaneous.

Solve each system by graphing.

15. $\begin{cases} x + y = -5 \\ 3x - y = 9 \end{cases}$

16. $\begin{cases} 3x - 2y = 9 \\ 2x - 3y = 2 \end{cases}$

17. $\begin{cases} y \geq 5 \\ x + y < 10 \end{cases}$

Solve each system by substitution.

18. $\begin{cases} y = -2x \\ 2x + 5y = 10 \end{cases}$

19. $\begin{cases} 4x + 5y = -1 \\ 3x - 2y = 5 \end{cases}$

20. $\begin{cases} 2a + 3b = 2 \\ a - 2b = 8 \end{cases}$

Solve each system by addition.

21. $\begin{cases} x - 2y = -13 \\ x + 2y = 3 \end{cases}$

22. $\begin{cases} 4x + 5y = 7 \\ 5x + 6y = 8 \end{cases}$

23. $\begin{cases} 2x - 3y = 11 \\ 4x = 6y + 15 \end{cases}$

24. Two girls went to the store for rice and sugar. One girl bought 3 pounds of sugar and 2 pounds of rice for 47 cents. The other girl bought 2 pounds of sugar and 2 pounds of rice for 39 cents. Find the price of a pound of sugar and of a pound of rice.

25. The sum of the numbers named by the digits in a two-digit numeral is 12. The number named by the numeral is 18 more than the number named when the digits are reversed. Find the number.

CHAPTER TEST

1. Which ordered pair (a, b) is in the solution set of the following system?

$$\begin{cases} 3a + b = 10 \\ b = 8 - 2a \end{cases}$$

a. (4, −2) **b.** (2, −4) **c.** (−2, −4) **d.** (2, 4)

2. Which systems of equations are equivalent to the following system?

$$\begin{cases} 3x - 5y - 19 = 0 \\ 2x + y - 4 = 0 \end{cases}$$

a. $\begin{cases} 3x - 5y - 19 = 0 \\ (3x - 5y - 19) + (2x + y - 4) = 0 \end{cases}$

b. $\begin{cases} 2x + y - 4 = 0 \\ x - 4y - 23 = 0 \end{cases}$

c. $\begin{cases} 3x - 5y - 10 = 0 \\ 10x + 5y - 20 = 0 \end{cases}$

d. $\begin{cases} x = 3 \\ 2x + y - 4 = 0 \end{cases}$

e. $\begin{cases} x = 3 \\ y = 2 \end{cases}$

3. In Exercise 2, what is the solution set of $\begin{cases} 3x - 5y - 19 = 0 \\ 2x + y - 4 = 0 \end{cases}$?

4. Solve graphically. $\begin{cases} 2x - y = -8 \\ x + y = -1 \end{cases}$

5. Solve by substitution. $\begin{cases} y = 2x + 2 \\ 4x - y = 6 \end{cases}$

6. Solve by addition. $\begin{cases} \\ 5a - 7b = 45 \end{cases}$

Solve by any method you wish.

7. $\begin{cases} p + 3q = 8 \\ 2p - 10q = -40 \end{cases}$

8. $\begin{cases} x + 3y = 6 \\ 5x + 7y = -2 \end{cases}$

9. Draw the graph of the system $\begin{cases} y \geq x + 5 \\ x + y \leq 8 \end{cases}$. Write the coordinates of three ordered pairs of numbers in the solution set.

10. The sum of two numbers is 56; their difference is 14. Find each number.

ENRICHMENT

A Computer Flow Chart for Linear Equations

Computers of all sizes and varying degrees of complexity are becoming everyday tools in many areas of life. Schools use computers to set up teaching schedules and examination schedules. Computers are used to play chess and checkers. They figure out such things as telephone bills and insurance premiums. And they are used in nearly all advanced science and space technology.

After the machines have been designed and built and before they can provide useful information, a set of instructions, a **program,** must be drawn up telling the machine, step by step, how to solve a given problem. Computers are constructed in such a way that they are able to do the basic processes of arithmetic: addition, multiplication, subtraction, division, finding roots, and finding absolute values. Computers can also compare different numbers by answering *Yes* or *No* to simple questions about relative magnitudes. For example, you may ask a computer if 5 is greater than 3; it will compute $5 - 3 = 2$, and answer *Yes*. It compares two numbers by subtracting.

The program given to a computer to enable it to solve a problem must be broken down into these simple operations. Each step of the program must involve only one operation and no steps can be omitted.

When the computer has been given a program, it operates on numbers that are stored in various numbered locations, such as [1], [17], and [*n*], in the machine until they are needed. And there are empty locations in which the machine may store any results it arrives at.

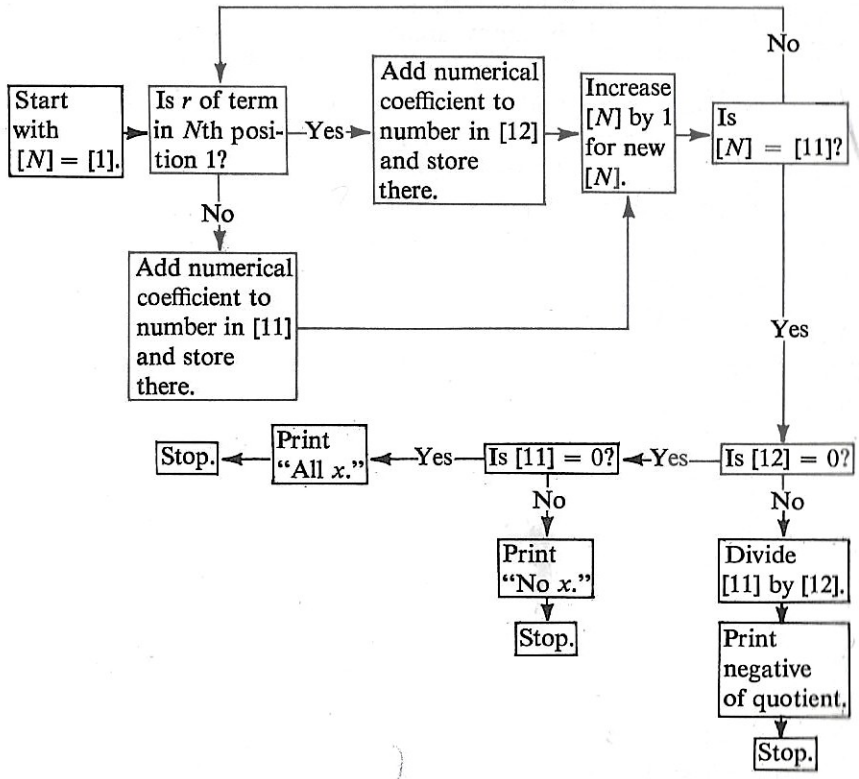
The final answer is printed on tape or on sheets of paper and the problem is solved, all in a matter of seconds.

The person who draws up the program, the programmer, usually sets it up in a diagram as shown below. It is called a **flow chart.** The arrows indicate the step-by-step process that the program follows.

Suppose you want to solve a linear equation in one unknown. Assume you have a computer that can handle such equations with up to ten terms, that is, $ax^r + bx^r + cx^r + dx^r + ex^r + fx^r + gx^r + hx^r + ix^r + jx^r$, $r = 0$ or 1. ($x^1 = x$; $x^0 = 1$; thus, $ax^0 = a$.) The terms ax^r through jx^r are assigned locations [1] through [10], respectively. All other locations in the computer begin empty.

To solve the equation, the computer is given the instructions in the flow chart below. It proceeds from box to box, following the appropriate path if the directions require a *Yes* or *No* answer. [*N*] stands for the number of the location. [11] is the location for constants, [12] for coefficients of the x term. Notice that the directions in one box read "Is [12] = 0?" This means "Is *the number in* [12] = 0?"

Flow Chart for Solving Linear Equation



You can begin by making a simple table, showing the location of each term and putting each term in ax^r form.

Location	1	2	3	4	5	6	7	8	9	10	11	12
Term	$+4x^1$	$+3x^0$	$-2x^1$	$+4x^1$	$-5x^0$	$-6x^1$	$+1x^0$	$+1x^1$	$-3x^1$	$+7x^0$		

By following the chart, you reason as follows.

Start with $[N] = [1]$; the first term is $+4x^1$. Is $r = 1$? Yes. Add 4, the coefficient, to the number in [12] ([12] is empty until now) and store it there. Number in [12] is 4. Increase $[N]$ by 1; so $[N]$ is [2]. Is $[N] = [11]$? No. Go back to the second box. Is r of $+3x^0$ equal to 1? No. Add $+3$ to the number in [11]. Store $+3$ in [11]. Increase $[N]$ by 1. Is $[N] = [11]$? No, $[N]$ is now [3]. Go back to the second box. Is r of $-2x^1$ equal to 1? Yes. Add -2 to 4, which is in [12]. [12] is now 2. Increase $[N]$ by 1. Is $[N] = [11]$? No, $[N]$ is 4. Go back to the second box and continue through $[N] = [10]$.

After you do the term $+7x^0$ and add 1 to $[N]$, $[N]$ is [11]. Follow the *Yes* path. Is [12] = 0? No, [12] is -2; [11] should be $+6$. Divide $+6$ by -2. The quotient is -3. Print the negative of -3, or $+3$. Stop. The problem is solved; x is 3.

You can check the computer's work by using algebra.

$$\begin{aligned} 4x + 3 - 2x + 4x - 5 - 6x + 1 + x - 3x + 7 &= 0 \\ (4x - 2x + 4x - 6x + x - 3x) + (3 - 5 + 1 + 7) &= 0 \\ -2x + 6 &= 0 \\ -2x &= -6 \\ x &= 3 \end{aligned}$$

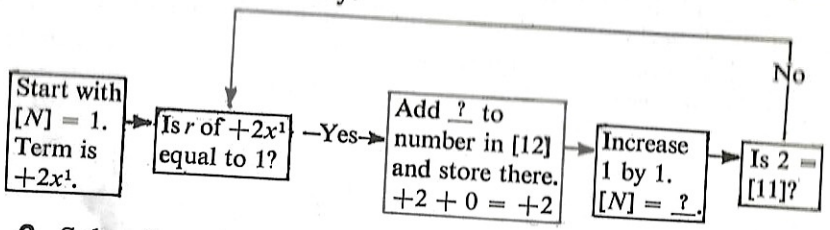
You may think that it is faster to solve such an equation by algebra instead of by following the flow chart, and for you it is. But the computer works so quickly that all the steps are accomplished in a number of seconds. Its use is so widespread because it is a great timesaving device and because it can perform the same operation over and over again in exactly the same way.

Exercises

1. Solve $2x + 1 - 4x + 5 - 3 + 3x - 2x + 4 - x + 1 = 0$. Begin by completing the following table as in the example.

Location	1	2	3	4	5	6	7	8	9	10	11	12
Term	$+2x^1$	$+1x^0$										

Then follow the flow chart on page 251. The path for the first term would be as shown below. Follow each step, filling in the blanks where necessary.



2. Solve $3x + 4 + 2x - 3 - x - 4 = 0$ by following the flow chart.

3. What did you do in Exercise 2 after the term 4? The rest of the terms should have been $0x^0$. So for $N = 7$, r was 0, and you added 0 to the number in [11]. For $N = 8$, r was 0, and again you added 0 to the number in [11]; and so on for $N = 9$ and $N = 10$. You performed four cycles that added nothing to the solution of the equation.

Place instructions in the program to eliminate excess cycles. You must do this by using questions that require *Yes* or *No* answers. The answers should be obtained by no more than the basic operations of arithmetic and comparing numbers.

4. Solve $3x + 5 - 8x + 4 - 9 + 5x = 0$, using the program with the additional instructions.

THE GROWTH OF MATHEMATICS

Greek Mathematics Through Diophantus

The Greeks are indebted to the Egyptians and the Orientals for the foundations of their mathematics. But whereas the Egyptians were very practical people, the Greeks were philosophical. In Greek society, manual and mechanical labor was done by slaves and was looked down upon by the leisure upper classes, who devoted their time to intellectual pursuits. Thus, mathematics changed greatly with the Greeks and became an abstract science.

One of Pythagoras' great contributions to mathematics was realizing that any postulational system had to begin with some assumptions. Then the rest of the system could be proven from the given assumptions.

as the head of the Pythagorean school in Tarentum. He advanced the theory of proportion and was one of the first to solve the ancient problem of duplicating the cube, that is, finding the side of a cube whose volume is double that of a given cube. He also advanced geometry by going against the rules of the Pythagoreans and applying it to mechanics: he supposedly worked out the theory of the pulley. When he was drowned in a shipwreck, the more rigid Pythagoreans thought it a fitting end for someone who had profaned pure mathematics by this application.

Philolaus, in the fifth century B.C., was another outstanding Pythagorean. He was the first to write a book on the Pythagorean teachings and revealed what had been kept secret for about a century.

Zeno of Elea (495–435 B.C.) greatly disturbed the philosophers of Athens by proposing four paradoxes involving infinitely small units of time or distance, the best known being the Achilles paradox: Achilles can run 10 times as fast as the tortoise he is going to race; if the tortoise is given a headstart of, say, 1000 yards, Achilles will never overtake him; for when Achilles has run 1000 yards, the tortoise is still 100 yards ahead of him; when Achilles has run that 100 yards, the tortoise is still 10 yards ahead of him; and so on. The fallacy in Zeno's reasoning was assuming that a finite distance can be divided into an infinite number of parts. But he had injected doubt into Greek thought, and the Greeks were ever suspicious of dealing with infinitely small quantities, or infinitesimals.

Plato (427?–347 B.C.), one of the greatest philosophers of antiquity, was not a mathematician in his own right, but he has been called "the maker of mathematicians" because he irritated so many into creating mathematics. His protegé was Eudoxus (408–355 B.C.), who developed the method of exhaustion for finding areas to avoid dealing with infinitesimals. [Think of a square inscribed in a circle (the four vertices touch the circle). Then think of an inscribed polygon with five sides, ten sides, twenty sides, and so on. The polygon more and more closely approximates the circle, and the area of the polygon is a good approximation for the area of the circle.] He also defined ratio sufficiently to allow mathematicians to work with irrational numbers as rigorously as they work with rational numbers.

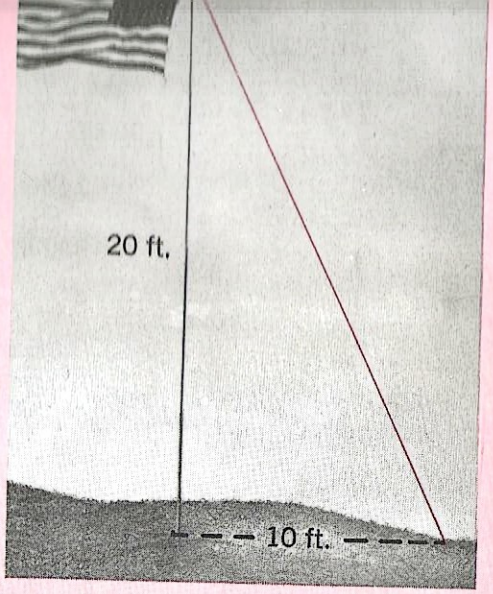
Euclid (330?–?275 B.C.) is usually connected with geometry. But of the thirteen books of his *Elements*, a majority deal entirely or in part with number theory and algebra. Much of the work is probably not his own, but rather a compilation of the work of his predecessors, and as such he did mathematics a great service. The *Elements* was a

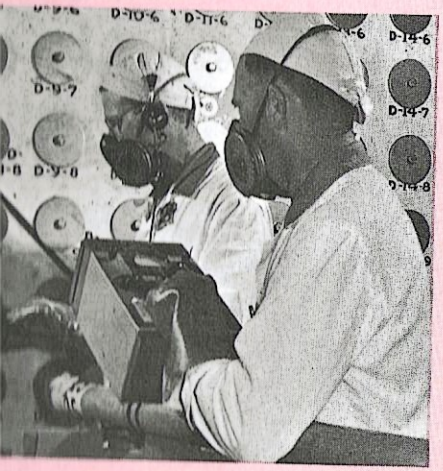
sand years. A follower in the Euclidean tradition is Archimedes (287?–212 B.C.), who will be treated in a separate section.

After the death of Euclid, the Greeks were occupied with geometry, and arithmetic and algebra were neglected. One man did make a contribution during this time and that was Eratosthenes (275–194 B.C.) who devised the "sieve" for finding prime numbers.

Diophantus of Alexandria, who flourished around A.D. 250, wrote a treatise on algebra, using symbolism and abbreviations that were a major advance at the time. Although he solved linear, quadratic, and cubic equations and could solve equations simultaneously, he failed to recognize that an equation could have more than one solution and would not accept negative or irrational solutions. Thus, equations with positive integral solutions are called Diophantine equations. Although he did not seem to care about the general solutions of equations, Diophantus' real genius lay in the fact that he could reduce almost any problem to one of the few types of equations that he could solve.

Greek mathematics then declined, and when Alexandria fell to the Arabs in A.D. 641, the glorious age of learning was over. The following centuries in Europe were called the Dark Ages, and only because of the monks and a few learned men was Greek culture preserved at all.



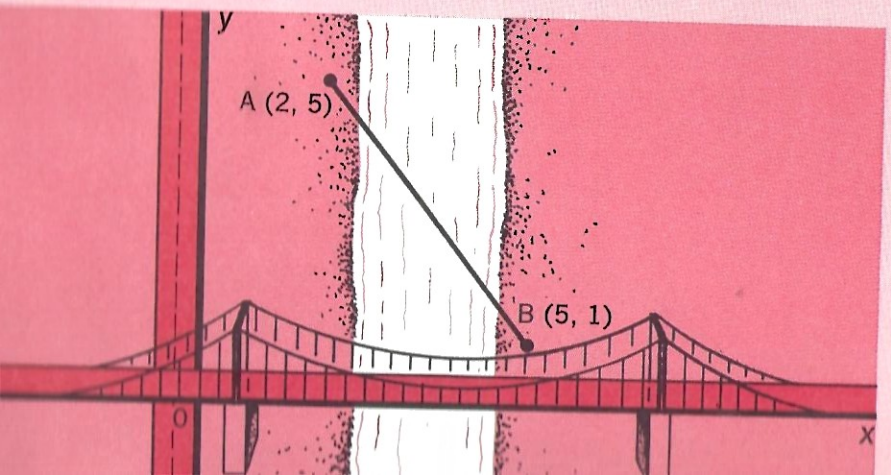


$\sqrt{5}$ $\quad \frac{1}{\sqrt{5}}$

$\sqrt[3]{5}$ $\quad \frac{1}{5^{\frac{1}{2}}}$

5^{-2} $\quad 5^{\frac{1}{2}}$

$5^{-\frac{1}{2}}$ $\quad 5^{\frac{1}{3}}$



CHAPTER 6

EXPONENTS AND RADICALS

The use of exponents and radicals is an important method of representing numbers that has many applications, including the distance formula for the coordinate plane. The interrelationships between exponents and radicals are useful in many branches of mathematics as well.

● A radioactive substance, as shown in the center left photograph, loses mass at a constant rate. Suppose that the substance shown loses half its mass every hour. In four hours, a mass of m will be reduced to $m(\frac{1}{2})^4$, read "m times one-half to the fourth power." If $(\frac{1}{2})^4$ means $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$, how much of the mass will be left at the end of four hours?

● Although Alpha Centauri (upper left photograph) is the nearest bright star to earth other than the sun, it is approximately 2.5×10^{13} miles away. This may not seem very far until you recall that 10^{13} means 10 used as a factor 13 times. Carrying out the multiplication gives 25,000,000,000,000. Why are exponents used in expressing such large numbers?

● By the Pythagorean theorem, the wire drawn in the upper right picture is $\sqrt{500}$ feet long. If you were buying wire, would you ask for $\sqrt{500}$ feet? What rational number could you give as an approximation for $\sqrt{500}$?

● In the lower drawing, points A and B are located with coordinates, using the bridge and the road as axes. How long is the cable across the river? Can you find the distance between any two points whose coordinates you know?

● In the center, can you match each expression in the left column with an expression in the right column? How are negative and fractional exponents defined?

6–1 Exponents and Multiplication

You know that 5^2 means $5 \cdot 5$ and 3^4 means $3 \cdot 3 \cdot 3 \cdot 3$. The ^{2} in 5^2 and the ^{4} in 3^4 are **exponents,** which show how many times the **base** (5 in 5^2 and 3 in 3^4) is taken as a factor. The expression 5^2 may be read as "five squared" or as "five to the second **power.**" The expression 3^4 is read as "three to the fourth power."

Definition If a is a positive integer, then x^a means

$$\underbrace{x \cdot x \cdot x \cdot x \cdot \cdots \cdot x}_{a \text{ factors}}$$

when a is greater than 1. Also, $x^1 = x$ is true.

One way to find the product of such expressions as 3^5 and 3^2 is to name 3^5 as 243 and 3^2 as 9 and then multiply in the ordinary way to get 2187. Another way to find the product is to use exponents.

$$\begin{aligned} 3^5 \cdot 3^2 &= (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3)(3 \cdot 3) \\ &= 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^7 \end{aligned}$$

The product named in exponent form is 3^7 or in decimal form is 2187.

How is the definition of a **positive integral exponent** used in **a–d**?

a. $n^3 \cdot n^5 = (n \cdot n \cdot n)(n \cdot n \cdot n \cdot n \cdot n) = n^8$

b. $x^1 \cdot x^4 = x(x \cdot x \cdot x \cdot x) = x^5$

c. $n \cdot n^3 \cdot n^4 = n(n \cdot n \cdot n)(n \cdot n \cdot n \cdot n) = n^8$

d. $-x^5 \cdot x^3 = -1(x \cdot x \cdot x \cdot x \cdot x)(x \cdot x \cdot x) = -x^8$

In **d**, note that $-x^5$ means the negative of x^5, not the fifth power of $-x$, which would be represented by $(-x)^5$.

Do you see a pattern in finding products with exponents? Can you make a conclusion about a method for naming the product of two numbers in exponential form with the same base? When m and n are positive integers, you can name $a^m \cdot a^n$ as follows.

$$\begin{aligned} a^m \cdot a^n &= (\underbrace{a \cdot a \cdot a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})(\underbrace{a \cdot a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}) \\ &= \underbrace{a \cdot a \cdot a \cdot a \cdot \cdots \cdot a}_{m+n \text{ factors}} \\ &= a^{m+n} \end{aligned}$$

Thus, by the definition of a positive integral exponent, $a^m \cdot a^n = a^{m+n}$ is true. This proves the following theorem, stated first in words and then in symbols.

Theorem 6–1 To name the product of two numbers represented with exponents and the same base, add the exponents and use the same base.

$$\forall a \in R \quad \forall m \in N \quad \forall n \in N \quad a^m \cdot a^n = a^{m+n}$$

You can now perform the operations in **a–d** on the previous page in a more efficient way.

a′. $n^3 \cdot n^5 = n^{3+5} = n^8$

b′. $x^1 \cdot x^4 = x^{1+4} = x^5$

c′. $n \cdot n^3 \cdot n^4 = n^{1+3+4} = n^8$ (Remember that n is n^1.)

d′. $-x^5 \cdot x^3 = -1(x^{5+3}) = -x^8$

Now try these

Use the definition of a positive integral exponent to name these products in exponential form.

1. $x^8 \cdot x^2$ **2.** $4^5 \cdot 4^3$ **3.** $9 \cdot 9^8$

4. $x^{17} \cdot x^3$ **5.** $x^{100} \cdot x^{100}$ **6.** $-a^6 \cdot a^3$

Answers: **1.** x^{10} **2.** 4^8 **3.** 9^9 **4.** x^{20} **5.** x^{200} **6.** $-a^9$

Exercises

A Name the following products using exponential form.

1. $(x \cdot x \cdot x \cdot x)(x \cdot x \cdot x)$ **2.** $(3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3)$

3. $-1(m \cdot m \cdot m)(m)$ **4.** $x^4 \cdot x^5$ **5.** $x^2 \cdot x^4$

6. $m \cdot m^6$ **7.** $a^2 \times a^3$ **8.** $n^3 \times n^4$

9. $b \times b^2$ **10.** $a(a)$ **11.** $b^3 \times b$

12. $b^2 \times b^3$ **13.** $-b(-b)$ **14.** $b(-b)$

15. $-a^5(a^2)$ **16.** $x(-x^2)$ **17.** $-y^9(y^9)$

18. $-y^2(-y^2)$ **19.** $a(a^2)(a^3)$ **20.** $p^8 \times p^8$

21. $-x(-x^2)$ **22.** $3^6 \cdot 3^4$ **23.** $5(5^4)$

24. $2(2^3)(2^4)$ **25.** $3(3^3)(3^4)$ **26.** $5^9(-5^9)(5^9)$

— Name the following products using exponential form.

Example. $(4 \times 10^6)(2 \times 10^8) = (4 \times 2) \times (10^6 \times 10^8)$
$= 8 \times 10^{14}$

27. $(2 \times 10^9)(3 \times 10^4)$ **28.** $(7 \times 10)(9 \times 10^5)$

29. $(1.5 \times 10^3)(2 \times 10^6)$ **30.** $(2.4 \times 10^7)(3.9 \times 10)$

31. $(2.8 \times 10^9)(a \times 10^2)$ **32.** $(a \times 10^x)(b \times 10^y)$

B — Find the products in exponential form. All exponents are considered to be positive integers.

33. $x^a \cdot x^2$ **34.** $y^2 \cdot y^b$ **35.** $x^a \cdot x^b$

36. $a^{x-2} \cdot a^5$ **37.** $x^{3a} \cdot x^{2a}$ **38.** $(x^{2a+7})(x^{2a-8})$

39. $3^x \cdot 3^2$ **40.** $2^{4a} \cdot 2^{3a}$ **41.** $(2^{2a+7})(2^{7a-6})$

C — Find the products in exponential form. All exponents are considered to be positive integers.

42. $(x+2)^2 \cdot (x+2)^4$ **43.** $(x+2)^a \cdot (x+2)^6$

44. $(x+2)^{a+5} \cdot (x+2)^{3a-2}$ **45.** $(x+y)^{a+b} \cdot (x+y)^{a-b}$

46. $(2a-3b)^{3a-2b} \cdot (2a-3b)^{6a+10b}$ **47.** $(3x+y)^{4a+1} \cdot (3x-y)^{2-4a}$

6–2 Exponents and Raising to a Power

Often you will need to raise products, quotients, and powers to a power. Explain each of the following.

a. $(2 \cdot 3)^4 = (2 \cdot 3)(2 \cdot 3)(2 \cdot 3)(2 \cdot 3)$
$= (2 \cdot 2 \cdot 2 \cdot 2)(3 \cdot 3 \cdot 3 \cdot 3)$
$= (2^4)(3^4)$

b. $(-x)^6 = (-x)(-x)(-x)(-x)(-x)(-x)$
$= (-1)(-1)(-1)(-1)(-1)(-1)(x \cdot x \cdot x \cdot x \cdot x \cdot x)$
$= (-1)^6 x^6$

c. $(xy)^3 = xy \cdot xy \cdot xy = (x \cdot x \cdot x)(y \cdot y \cdot y)$
$= x^3 y^3$

Can you find a pattern? Does it lead to the following theorem?

Theorem 6–2 The positive integral power of a product is the product of the powers of the factors.

$$\forall x \in R \quad \forall y \in R \quad \forall n \in N \quad (xy)^n = x^n y^n$$

The proofs of the theorems in this section will be left as exercises.

In **b** the expression $(-1)^6$ occurred. Do you know how to raise -1 to any positive integral power? What previous generalization tells you that $(-1)^n$ is equal to -1 when n is odd and to 1 when n is even? Then **b** could be rewritten as follows.

b′. $(-x)^6 = (-1)^6 x^6 = 1 \cdot x^6 = x^6$

Can you explain the following?

d. $(-5)^3 = (-1)^3(5)^3 = (-1)(5)^3$
$= -125$

e. $(-a)^4 = (-1)^4 a^4 = 1 \cdot a^4$
$= a^4$

f. $(-t)^7 = (-1)^7 t^7 = (-1)t^7$
$= -t^7$

Theorem 6–3 An even positive power of a negative number is positive and an odd integral power of a negative number is negative.

The rules for multiplication of numbers expressed as fractions provide another way to rewrite expressions involving exponents.

g. $\left(\frac{3}{4}\right)^3 = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{3 \cdot 3 \cdot 3}{4 \cdot 4 \cdot 4}$
$= \frac{3^3}{4^3}$

h. $\left(\frac{x}{y}\right)^4 = \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} \cdot \frac{x}{y} = \frac{x \cdot x \cdot x \cdot x}{y \cdot y \cdot y \cdot y}$
$= \frac{x^4}{y^4}$

Theorem 6–4 A positive integral power of a number expressed as a fraction is equal to the power of the numerator divided by the power of the denominator.

$$\forall x \in R \quad \forall y \in R\ (y \neq 0) \quad \forall n \in N \quad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

Taking a positive integral power of a power can be viewed in two ways. You can go back to the basic definition of a positive integral exponent or you can use the theorem concerning powers of a product.

$(x^2)^3 = x^2 \cdot x^2 \cdot x^2$
$= x^{2+2+2}$
$= x^6$

$(x^2)^3 = (x \cdot x)^3$
$= x^3 \cdot x^3$
$= x^{3+3}$
$= x^6$

Explain each of the following.

i. $(5^4)^3 = 5^4 \cdot 5^4 = 5^{4+4} = 5^8$

j. $(t^3)^6 = (t \cdot t \cdot t)^6 = t^6 \cdot t^6 \cdot t^6 = t^{6+6+6} = t^{18}$

k. $(r^a)^4 = r^a \cdot r^a \cdot r^a \cdot r^a = r^{a+a+a+a} = r^{4a}$, where a is a positive integer.

l. $(b^4)^c = (b \cdot b \cdot b \cdot b)^c = b^c \cdot b^c \cdot b^c \cdot b^c = b^{4c}$, where c is a positive integer.

Theorem 6–5 When a positive integral power is raised to a positive integral power, the base is raised to the product of the two powers.

$$\forall x \in R \quad \forall m \in N \quad \forall n \in N \quad (x^m)^n = x^{mn}$$

The theorems of this section can be combined to give additional results.

m. $\left(\frac{x^5}{y^7}\right)^4 = \frac{(x^5)^4}{(y^7)^4} = \frac{x^{5 \cdot 4}}{y^{7 \cdot 4}} = \frac{x^{20}}{y^{28}}$

n. $(x^3y^2)^5 = (x^3)^5 \cdot (y^2)^5 = x^{15}y^{10}$

Exercises

A — Name each of the following in a simpler exponent form, so that each base is used only once and no power is raised to a power. The first example of each type is done for you. Variables in denominators may not represent 0.

1. $x^2(x^3) = x^5$
2. $3x(2x^2)$
3. $x^2(x^5)$
4. $(x^2y^2)(xy)(x^3y^2)$
5. $3x^2y(y^2)$
6. $y^4(y)$
7. $3y^2(y^2)$
8. $2(2^4)$
9. $2x(2^3)(x^4)$
10. $(ab)^3 = a^3b^3$
11. $(xy)^2$
12. $(2x)^2$
13. $(5x)^2$
14. $(2xy)^2$
15. $(xyz)^4$
16. $(-5)^2 = (-5)(-5) = 25$
17. $(-6)^2$
18. $(-3)^3$
19. $(-x)^4$
20. $(-x)^5$
21. $(-y)^2$
22. $(-4x)^2$
23. $(-3b)^3$
24. $(-ab)^3$
25. $\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$
26. $\left(\frac{x}{y}\right)^2$
27. $\left(\frac{1}{2}\right)^2$
28. $\left(\frac{2}{3}\right)^3$
29. $\left(\frac{1}{3}\right)^4$
30. $\left(\frac{c}{d}\right)^7$
31. $\left(\frac{x}{y}\right)^3$
32. $\left(\frac{2}{3}\right)^5$
33. $\left(\frac{ab}{c}\right)^5$
34. $(x^3)^2 = x^6$
35. $(x^3)^6$
36. $(x^4)^3$
37. $(y^5)^3$
38. $(a^3)^4$
39. $(x^3)^5$
40. $(x^7)^3$
41. $(x^5)^2$
42. $(x^3)^{19}$
43. $(x^2)^5$
44. $(y^3)^3$
45. $(a^5)^6$
46. $(b^4)^6$
47. $(a^2)^7$
48. $(y^4)^8$
49. $(3a^2b)^2 = 3^2a^4b^2 = 9a^4b^2$
50. $(6a^2)^3$
51. $(2x^5)^2$
52. $(-3b)^3$
53. $\frac{(2x^2)^3}{(3y)^2}$
54. $(3^2b^2)^3$
55. $(2x^3y^2)^3$
56. $(3^2a^2b^4)^2$
57. $(5x^2y^3)^2$
58. $(3x^2)^3$
59. $(4x^3y^2)^3$
60. $(2m^3n^2)^4$
61. $(10a^5b^2)^3$
62. $(7a^2b^3)^3$
63. $(x^2y)^4$
64. $(2a^2b)^3$
65. $(2^3x^2y^3)^4$
66. $(3a^3b^3)^4$
67. $(4a^4c^2d^3)^3$
68. $(16x^3y^2)^2$
69. $(3x^3y)^3$
70. $(2^3a^2b^4)^2$
71. $(2x^3y^2)^6$
72. $(a^2b)^3$
73. $(a^3b^2)^2$
74. $(-x^2y^3)^3$
75. $(-x^4y^3)^2$
76. $(-x^4y^2)^3$
77. $(2 \times 10^3)^2$
78. $(3 \times 10^5)^4$
79. $(1.1 \times 10^6)^3$
80. $(2.5 \times 10^{10})^2$
81. $\left(\frac{x^2}{y^3}\right)^2 = \frac{x^4}{y^6}$
82. $\left(\frac{x^2}{y^2}\right)^4$
83. $\left(\frac{b^2}{a}\right)^2$
84. $\left(\frac{c^2}{d}\right)^3$
85. $\left(\frac{a^2}{b^3}\right)^3$
86. $\left(\frac{a^3}{b}\right)^6$
87. $\left(\frac{x^3}{y^2}\right)^2$
88. $\left(\frac{c^3}{d^2}\right)^7$
89. $\left(\frac{a^4}{b^3}\right)^2$
90. $\left(\frac{2^2}{3^2}\right)^3$

B

91. $\left(\frac{5x^2y^3}{-3a^2b^3}\right)^2$
92. $(-2x^2)^2(2xy^2)^3$
93. $(-2a^3b^2)(3a^4b^6)$
94. $x \cdot x^2(x^3)^2$
95. $(-\frac{1}{3})^2(-\frac{1}{3})^3(-\frac{1}{3})^4$
96. $(-2)^2(-2)^3$
97. $(-3 \cdot 5x^2)^3 \cdot 2^4$
98. $\left(\frac{-3x^2}{4y^3}\right)^3$

In the following exercises, variables used in exponents represent positive integers. Variables in denominators may not represent 0. Rewrite each expression in a simpler form.

99. $(x^3)^n$	**100.** $(y^8)^n$	**101.** $(a^7)^x$
102. $(a^n)^m$	**103.** $(x^n)^n$	**104.** $(y^k)^c$
105. $(a^0)^m$	**106.** $(b^c)^0$	**107.** $(c^2)^w$
108. $(k^m)^t$	**109.** $\left(\frac{m}{n}\right)^k$	**110.** $\left(\frac{x^2}{y}\right)^c$
111. $\left(\frac{a^3}{b}\right)^d$	**112.** $\left(\frac{x^4}{y^a}\right)^b$	**113.** $\left(\frac{x^c}{y^z}\right)^5$
114. $\left(\frac{e^3}{f^2}\right)^x$	**115.** $\left(\frac{g^3}{h^4}\right)^m$	**116.** $\left(\frac{3m}{2n}\right)^c$
117. $\left(\frac{3x^n}{5y^m}\right)^c$	**118.** $\left(\frac{3x^3}{4y^2}\right)^m$	**119.** $\left(\frac{3a^mb^n}{2xy}\right)^k$
120. $\left(\frac{2ab}{4xy}\right)^d$	**121.** $\left(\frac{a^2b^2}{x^2y^3}\right)^b$	**122.** $((x^n)^n)^n$

C **123.** Prove Theorem 6–2.
124. Prove Theorem 6–3.
125. Prove Theorem 6–4.
126. Prove Theorem 6–5.

6–3 Exponents and Division

Does the number $\frac{2^5}{2^3}$ have a simpler name? By definition,

$$\frac{2^5}{2^3} = \frac{2\cdot2\cdot2\cdot2\cdot2}{2\cdot2\cdot2} = 2\cdot2 = 4$$

is true. Another way to say this is that $\frac{2^5}{2^3} = 2^2$ is true. Can you use the same method to explain **a**–**c** below?

a. $\frac{3^6}{3^2} = \frac{3\cdot3\cdot3\cdot3\cdot3\cdot3}{3\cdot3} = 3^4$ **b.** $\frac{2^8}{2} = 2^7$ **c.** $\frac{4^{10}}{4^2} = 4^8$

The same principle applies to variables, provided that the variable is not equal to zero. Explain each step in **d**–**f**.

d. $\frac{x^5}{x^3} = \frac{x\cdot x\cdot x\cdot x\cdot x}{x\cdot x\cdot x} = \frac{x\cdot x\cdot x}{x\cdot x\cdot x}\cdot x\cdot x = x^2,\ x \neq 0$

e. $\frac{x^6}{x^4} = x^2,\ x \neq 0$ **f.** $\frac{x^6}{x^5} = x,\ x \neq 0$

More complex expressions can be simplified in the same way. In **g**–**i**, none of the variables may represent 0.

g. $\frac{x^4y^3}{x^2y} = \frac{x\cdot x\cdot x\cdot x\cdot y\cdot y\cdot y}{x\cdot x\cdot y} = x\cdot x\cdot y\cdot y$
$= x^2y^2$

h. $\frac{a^3b^4c^3}{a^2b^3c} = \frac{a\cdot a\cdot a\cdot b\cdot b\cdot b\cdot b\cdot c\cdot c}{a\cdot a\cdot b\cdot b\cdot c} = a\cdot b\cdot b\cdot c$
$= ab^2c$

i. $\frac{-6a^5b^3c^2}{2a^2b^2c} = \frac{-3\cdot2\cdot a\cdot a\cdot a\cdot a\cdot a\cdot b\cdot b\cdot b\cdot c\cdot c}{2\cdot a\cdot a\cdot a\cdot b\cdot b\cdot c}$
$= -3\cdot a\cdot a\cdot b\cdot c$
$= -3a^2bc$

You have probably noticed that there is a pattern in these examples. Can you put it in words?

It is easy to prove that a pattern exists. In the following, x cannot equal 0 and m is greater than n.

① Use definition of exponent.

$$\frac{x^m}{x^n} = \frac{\overbrace{x\cdot x\cdot x\cdot x\cdot\cdots\cdot x}^{m\text{ factors}}}{\underbrace{x\cdot x\cdot x\cdot x\cdot\cdots\cdot x}_{n\text{ factors}}}$$

② Divide numerator and denominator by x, repeated n times.

$$= \overbrace{x\cdot x\cdot x\cdot x\cdot\cdots\cdot x}^{(m-n)\text{ factors}}$$

③ Use definition of exponent.

$$= x^{m-n}$$

Theorem 6–6 To divide two numbers each with a positive integral exponent and with the same base, subtract the exponent of the denominator from the exponent of the numerator.

$$\forall x\in R\ (x \neq 0)\quad \forall m, n\in N\ (m > n)\quad \frac{x^m}{x^n} = x^{m-n}$$

Can you guess why the condition that m is greater than n is necessary? When n and m are the same, $m - n$ names 0; and when m is less than n, the expression $m - n$ names a negative number. So far, zero and negative exponents have not been defined. This will be done in the next section of this chapter.

Exercises

A — Simplify each of the following exponential expressions. Variables in the denominators cannot represent 0.

1. $\frac{2^8}{2}$ 2. $\frac{3^5}{3^3}$ 3. $\frac{5^6}{5^3}$

4. $\frac{x^5}{x^3}$ 5. $\frac{a^6}{a}$ 6. $\frac{a^9}{a^7}$

7. $\frac{x^8}{x^7}$ 8. $\frac{x^3y^2}{xy}$ 9. $\frac{a^3y^3}{a^2y}$

10. $\frac{y^3}{y^2}$ 11. $\frac{a^3b^5}{ab^2}$ 12. $\frac{2x^7}{3x^4}$

13. $\frac{k^5m^3}{k^3m^2}$ 14. $\frac{18x^3y^4}{2xy^2}$ 15. $\frac{-15x^7y^5}{-3x^2y^4}$

16. $\frac{-36a^2b^2c^4}{4abc^3}$ 17. $\frac{-27r^5s^8}{-3r^3s^6}$ 18. $\frac{-2k^4m^3}{4k^2}$

19. $\frac{1.5 \times 10^4}{2.5 \times 10^2}$ 20. $\frac{2.4 \times 10^7}{5.6 \times 10^4}$ 21. $\frac{8.01 \times 10^{11}}{6.12 \times 10}$

B — Perform the indicated operations to obtain an expression in a simple form. No variable in the denominator can be zero.

Example. $\frac{9xy(-4x^2y^3)}{-18x^2y^2} = \frac{-36x^3y^4}{-18x^2y^2} = 2xy^2$

22. $\frac{4ab(7a^2b^3)}{-14a^3b^3}$ 23. $\frac{4ab^2(-5ab^3)}{10a^2b^2}$ 24. $\frac{(-3ab^2)^2}{6ab}$

25. $\frac{(3ab^2)^2(-6a^3b)}{(3a^2b^2)^2}$ 26. $\frac{(5ab)^2(-20a^3b)}{4a^2b^3}$ 27. $\frac{(5a^2b)^2(-100b^3)}{(5^2b)^2}$

28. $\frac{(3.5 \times 10^2)^3}{1.5 \times 10^2}$ 29. $\frac{(7.5 \times 10^3)^4}{(2.5 \times 10^2)^3}$ 30. $\frac{(1.21 \times 10)^5}{(1.1 \times 10^2)^2}$

C — Perform the indicated operations. Variables in exponents are all positive integers.

31. $\frac{a^xb^2y}{a^xby}$ 32. $\frac{a^xb^4}{a^y}$ 33. $\frac{a^tb^w}{a^vb^{25}}$

34. $\frac{x^{u-v}y^{w-z}}{x^{2v-u}y^w}$ 35. $\frac{x^{3u-2v}y^3}{x^{2u+2v}y^2}$ 36. $\frac{a^{3x}}{a^{2x}}$

37. $\frac{a^{2x}b^{3y}c^k}{abc}$ 38. $\frac{ab^2c^4d^{2-k}}{ac^3d^k}$ 39. $\frac{(x^{2a+1})^2}{x^{a-1}}$

40. $\frac{(y^{3a-9})^6}{y^{2a-4}}$ 41. $\left[\frac{(x^{2a-4})^2}{x^{a+5}}\right]^3$ 42. $\frac{(1.69 \times 10^3)^a}{(1.3 \times 10^a)^2}$

6–4 Zero and Negative Integral Exponents

A positive integer as an exponent indicates the number of times the base is taken as a factor. You cannot apply the same meaning to cases such as a^0 or x^{-3}, for you cannot take a as a factor zero times or x as a factor negative three times. To use **zero and negative integral exponents,** you must give a new meaning to the idea of an exponent.

The definition of a zero exponent is chosen so that you can continue to use Theorem 6–6 $\left(\frac{a^m}{a^n} = a^{m-n}\right)$ when m and n are equal.

If $\frac{a^m}{a^n} = a^{m-n}$ is to be true, then when $m = n$	You know that	Then it is reasonable to define
$\frac{x^3}{x^3} = x^{3-3} = x^0, x \neq 0$	$\frac{x^3}{x^3} = 1, x \neq 0$	$x^0 = 1, x \neq 0$
$\frac{10^7}{10^7} = 10^{7-7} = 10^0$	$\frac{10^7}{10^7} = 1$	$10^0 = 1$
$\frac{a^m}{a^m} = a^{m-m} = a^0, a \neq 0$	$\frac{a^m}{a^m} = 1, a \neq 0$	$a^0 = 1, a \neq 0$

Definition Any nonzero base with an exponent of zero is equal to 1.

$$\forall a \in R\ (a \neq 0)\quad a^0 = 1$$

If a is 0, then a^0 is not defined. Is each of the following true?

a. $7^0 = 1$ **b.** $695^0 = 1$ **c.** $(-5\frac{1}{2})^0 = 1$

d. $(x^2y^3)^0 = 1, x \neq 0, y \neq 0$ **e.** $1^0 = 1$

The definition of a negative integral exponent is also chosen to be consistent with Theorem 6–6. In the table, x and a cannot be zero.

If $\frac{a^m}{a^n} = a^{m-n}$ is to be true, then when $m < n$	You know that	Then it is reasonable to define
$\frac{x^5}{x^8} = x^{5-8} = x^{-3}$	$\frac{x^5}{x^8} = \frac{x^5}{x^5 \cdot x^3} = \frac{1}{x^3}$	$x^{-3} = \frac{1}{x^3}$
$\frac{10^8}{10^{12}} = 10^{8-12} = 10^{-4}$	$\frac{10^8}{10^{12}} = \frac{10^8}{10^8 \cdot 10^4} = \frac{1}{10^4}$	$10^{-4} = \frac{1}{10^4}$
$\frac{a^5}{a^{5+m}} = a^{5-(5+m)} = a^{-m}$	$\frac{a^5}{a^{5+m}} = \frac{a^5}{a^5 \cdot a^m} = \frac{1}{a^m}$	$a^{-m} = \frac{1}{a^m}$

Definition Any nonzero base raised to a negative power is the multiplicative inverse of the base raised to the corresponding positive power.

$$\forall a \in R\ (a \neq 0)\quad \forall m \in I\quad a^{-m} = \frac{1}{a^m}$$

Explain why each of the following is true.

f. $x^{-5} = \frac{1}{x^5}$ **g.** $(\frac{1}{3})^{-2} = \frac{1}{(\frac{1}{3})^2} = 3^2$ **h.** $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

The use of zero and negative exponents gives you a new way to look at place value, one of the important characteristics of our numeration system. You know that a number such as 346.78 can be named as shown below.

$$346.78 = 3(100) + 4(10) + 6(1) + 7(\tfrac{1}{10}) + 8(\tfrac{1}{100})$$

You know that 10 and 100 are powers of ten and that each power is multiplied by one of the integers 0 through 9. Using the two new definitions concerning exponents in this section, you should recognize 1, $\frac{1}{10}$, and $\frac{1}{100}$ as powers of ten, also. Expressing a number as the sum of powers of ten multiplied by one of the integers 0 through 9 is known as representing the number in **expanded notation.**

$$346.78 = 3(10^2) + 4(10^1) + 6(10^0) + 7(10^{-1}) + 8(10^{-2})$$

The decimal point in a numeral indicates the units place and is placed just to the right of the units place.

Do you see the pattern? How do you determine the exponent of 10 for a digit 3 places to the right of the units place? 12 places to the left? n places to the right? n places to the left?

Exponents are also used in another way in representing numbers. In **scientific notation** a number is represented as a power of 10 multiplied by a number greater than or equal to 1 and less than 10. One advantage of scientific notation is that it is a compact way of representing numbers with a very large or very small absolute value. To indicate negative numbers in scientific notation, the multiplier should be a number less than or equal to -1 and greater than -10.

The following sentences indicate the usefulness of scientific notation. Explain each.

i. $24{,}000{,}000{,}000{,}000 = 2.4 \times 10^{13}$ **j.** $0.00000000368 = 3.68 \times 10^{-9}$

k. $346.78 = 3.4678 \times 10^2$ **l.** $-59{,}000{,}000 = -5.9 \times 10^7$

Now try these

— Name without using exponents.

1. 3^0 **2.** $(-6\frac{1}{5})^0$ **3.** $(4)^{\frac{1}{-2}}$

4. $(-4)^{-1}$ **5.** $(-\frac{2}{3})^{-3}$ **6.** $(263)^0 \cdot (5)^{-2}$

— Write in expanded notation.

7. 632.96 **8.** 0.00032 **9.** 3.00029

10. Rewrite Exercises 7 and 8 using scientific notation.

Answers: **1.** 1 **2.** 1 **3.** $\frac{1}{4^2} = \frac{1}{16}$ **4.** $\frac{1}{(-4)^1} = \frac{1}{-4} = -\frac{1}{4}$ **5.** $\frac{1}{(-\frac{2}{3})^3} = \frac{1}{-\frac{8}{27}} = -\frac{27}{8}$ **6.** $1 \cdot \frac{1}{5^2} = \frac{1}{25}$ **7.** $6(10^2) + 3(10^1) + 2(10^0) + 9(10^{-1}) + 6(10^{-2})$ **8.** $3(10^{-4}) + 2(10^{-5})$ **9.** $3(10^0) + 0(10^{-1}) + 0(10^{-2}) + 0(10^{-3}) + 2(10^{-4}) + 9(10^{-5})$ **10.** 6.3296×10^2; 3.2×10^{-4}

Exercises

A — Name without using exponents.

1. 5^{-2}	**2.** 10^{-2}	**3.** 2^{-3}	**4.** 8^0
5. 4^2	**6.** 3^{-2}	**7.** 10^0	**8.** 10^{-1}
9. 10^3	**10.** 10^{-4}	**11.** 6^0	**12.** $(-2)^3$
13. $(-2)^{-3}$	**14.** $(\frac{1}{2})^{-4}$	**15.** $(\frac{3}{4})^{-3}$	**16.** 157^0
17. $(\frac{1}{4})^{-2}$	**18.** $(-\frac{3}{4})^3$	**19.** $(-\frac{3}{4})^{-3}$	**20.** $(-4)^{-3}$
21. 10^{-3}	**22.** $(-\frac{1}{2})^{-2}$	**23.** $(-\frac{2}{3})^{-2}$	**24.** $(-\frac{1}{2})^{-5}$

— Name each number as a multiple of a power of 10, that is, in scientific notation.

25. 600	**26.** 50	**27.** $\frac{3}{100}$	**28.** 7000
29. 9000	**30.** 20	**31.** $\frac{5}{1000}$	**32.** $\frac{8}{10}$
33. 3000	**34.** 40,000	**35.** $\frac{2}{100}$	**36.** $\frac{4}{10{,}000}$

— Name each number in expanded form.

37. 5,000,000	**38.** 3600	**39.** 0.000027
40. 5,630,000	**41.** 342.968	**42.** 6829.6829
43. 349.6	**44.** 497.35	**45.** 4.732
46. 47.32	**47.** 473.2	**48.** 4732
49. 0.0043	**50.** 0.00057	**51.** 400.05
52. 0.000008	**53.** 50,000.04	**54.** 93,000,000

55. Name each of the numbers in Exercises 37–54 in scientific notation.

B **56.** Multiply each term of $(3 \times 10^3) + (2 \times 10^2) + (1 \times 10^1) + (5 \times 10^0) + (8 \times 10^{-1}) + (9 \times 10^{-2})$ by 10. Use your results to explain why multiplying 3215.89 by 10 has the effect of moving each digit one place to the left with reference to the decimal point.

57. Repeat Exercise 56, but multiply by 10^2, or 100. Multiplying 3215.89 by 100 has the effect of doing what to each digit?

58. Repeat Exercise 56, but use 10^3, or 1000.

Find the number named by each of the following if x is -1 and y is 2.

59. $(x^2)(y^0)$	**60.** $(x^{-2})(y^3)$	**61.** $(x^{-3})(y)$	**62.** $3x^2y^{-4}$
63. y^{-5}	**64.** $-2x^0y^{-2}$	**65.** $-3x^{-2}y^{-3}$	**66.** $-x^0y^{-6}$
67. $x^{-2} + y^{-2}$	**68.** $x^3 + y^{-3}$	**69.** $x^{-3} + y^{-3}$	**70.** $x^4 + y^{-4}$

C **71.** If $x > 0$, is x^{-9} positive, negative, or zero?

72. If $x < 0$, is x^9 positive, negative, or zero?

73. If $x < 0$, is x^{-8} positive, negative, or zero?

74. If $x < 0$, is x^{-9} positive, negative, or zero?

75. If $x > 0$ and m is an integer greater than zero, is x^m positive, negative, or zero?

76. If $x > 0$ and m is an integer less than zero, is x^m positive, negative, or zero?

77. If x is any nonzero real number and m is an even integer, is x^m positive, negative, or zero?

78. The definitions of zero and negative exponents were based upon Theorem 6–6. Prove that Theorems 6–1 and 6–5 are true for zero and negative exponents.

6–5 Roots of Numbers

Previously you have learned the following definition.

Definition A square root of a ($a \geq 0$) is a number b such that $b^2 = a$ is true.

The definition applies to nonnegative numbers only. If a is -25, could you find its square root? You could try -5, but $(-5)^2$ is 25.

According to the definition, -5 is a square root of 25, but not of -25. The number -25 has no real number as a square root.

One of the square roots of 25 is -5, as was seen in the last paragraph. You are already familiar with the other square root of 25, which is 5.

Every positive real number has two square roots.

For some numbers, for example, 2, the square roots are irrational. Can you tell which real number has just one square root?

The symbol $\sqrt{a}$ is used to denote the nonnegative square root of a ($a \geq 0$). The symbol $\sqrt{a}$ is a **radical** and $\sqrt{\ }$ is the **radical sign.** The number a is the **radicand.**

The sentence $5^3 = 125$ shows the operation of cubing 5. The inverse operation of cubing — taking the **cube root** of a number — is finding the solution of $125 = x^3$. Since $5^3 = 125$ is true, 5 is a cube root of 125. Is -5 a cube root also? The number $(-5)^3$ is equal to $(-5)(-5)(-5)$, or -125. Hence, 5 is the only real number whose cube is 125, and there is only one real number that makes $125 = x^3$ true.

Does -5 make the equation $-125 = x^3$ true? Does any other real number make the equation true? For negative numbers you can find a real number that is the cube root.

Definition The cube root of a is the number b such that $b^3 = a$ is true. The cube root of a may be written as the radical $\sqrt[3]{a}$.

In the radical sign for cube roots, the raised numeral 3 is the **index** of the radical. The index 2 for a square root is not usually written, so $\sqrt{a}$ means $\sqrt[2]{a}$.

Since, by convention, $\sqrt{a}$ denotes the nonnegative square root, the symbol $\sqrt{16}$ names only one number, 4. The other square root of 16 is $-\sqrt{16}$, or -4. Since there is only one real cube root, $\sqrt[3]{-8} = -2$ and $\sqrt[3]{125} = 5$. A radical always names just one number. This number is the **principal root.**

Definition The principal root of a number is the real root if there is only one. If there are two roots, the principal root is the positive one.

plus-or-minus sign. The two square roots of 16 may be written either as ± 4 or as $\pm\sqrt{16}$.

It is easy to extend the idea of roots and radicals to fourth roots, fifth roots, sixth roots, and so on.

$\sqrt[4]{16} = 2$ because $2^4 = 16$

$\sqrt[5]{32} = 2$ because $2^5 = 32$

$\sqrt[10]{1024} = 2$ because $2^{10} = 1024$

Definition If $a^n = b$, where n is a positive integer, then a is the *nth root* of b.

Now try these

— Name the two square roots of each number below.

1. 81
2. 16
3. $\frac{25}{36}$
4. 10,000
5. $\frac{9}{100}$
6. $\frac{10{,}000}{144}$

— Name the real cube root of each number below.

7. 8
8. -1
9. 1000
10. -216
11. $-\frac{8}{27}$
12. $(-17)^3$

— Express each number without using the radical sign.

13. $-\sqrt{36}$
14. $\sqrt[3]{-27}$
15. $\sqrt[4]{3^4}$
16. $\sqrt[7]{-128}$

Answers: 1. 9, −9 2. 4, −4 3. $\frac{5}{6}$, $-\frac{5}{6}$ 4. 100, −100 5. $\frac{3}{10}$, $-\frac{3}{10}$ 6. $\frac{100}{12}$, $-\frac{100}{12}$ 7. 2 8. −1 9. 10 10. −6 11. $-\frac{2}{3}$ 12. −17 13. −6 14. −3 15. 3 16. −2

Exercises

A — What number or numbers are named by each expression below? Name each number without using the radical sign.

1. $\sqrt{16}$
2. $-\sqrt{25}$
3. $\sqrt{49}$
4. $\pm\sqrt{144}$
5. $\pm\sqrt{100}$
6. $\pm\sqrt{81}$
7. $\sqrt{10{,}000}$
8. $\sqrt{5^2}$
9. $-\sqrt{4^2}$
10. $\pm\sqrt{8^2}$
11. $\pm\sqrt{100^2}$
12. $\sqrt{125^2}$
13. $\sqrt[3]{27}$
14. $\sqrt[3]{-64}$
15. $\sqrt[3]{125}$
16. $\sqrt[3]{-1000}$
17. $\sqrt[3]{\frac{1}{8}}$
18. $\sqrt[3]{-\frac{1}{64}}$
19. $\sqrt[3]{\frac{1}{125}}$
20. $\sqrt[3]{1{,}000{,}000}$

B — What number or numbers are named by each expression below? Name each number without using the radical sign.

21. $(\sqrt{2})^3$
22. $(\sqrt[3]{2})^3$
23. $(\sqrt[4]{2})^4$
24. $(\sqrt[5]{2})^5$
25. $(\sqrt{a})^3$
26. $(-\sqrt{2})^2$
27. $(-\sqrt[3]{2})^3$
28. $(\sqrt[9]{7})^9$
29. $(\sqrt[n]{2})^n$
30. $(\sqrt[n]{a})^n$
31. $(4\sqrt{5})^2$
32. $(-2\sqrt[3]{3})^3$
33. $(-2\sqrt[3]{3})^3$
34. $(5\sqrt{3})^{-2}$
35. $(8\sqrt{2})^0$
36. $5 \cdot (\sqrt{2})^0$
37. $(2\sqrt[3]{3})^{-3}$
38. $(3^0 \cdot \sqrt[4]{9})^4$
39. $(\sqrt[3]{-2})^3$
40. $(\sqrt[5]{-32})^3$

6–6 Rational Exponents

Study the table below to see if you can find a way to make a definition for exponents that are not integers, that is, **rational exponents.** The variables x and a represent nonnegative numbers.

If $x^a \cdot x^b = x^{a+b}$ and $(x^a)^b = x^{ab}$ are true for rational exponents, then	You know that	Then it is reasonable to define
$5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5^1 = 5$	$\sqrt{5} \cdot \sqrt{5} = 5$	$5^{\frac{1}{2}} = \sqrt{5}$
$2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 2^1 = 2$	$\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2} = 2$	$2^{\frac{1}{3}} = \sqrt[3]{2}$
$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$	$\sqrt{x} \cdot \sqrt{x} = x$	$x^{\frac{1}{2}} = \sqrt{x}$
$y^{\frac{1}{3}} \cdot y^{\frac{1}{3}} \cdot y^{\frac{1}{3}} = y^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = y^1 = y$	$\sqrt[3]{y} \cdot \sqrt[3]{y} \cdot \sqrt[3]{y} = y$	$y^{\frac{1}{3}} = \sqrt[3]{y}$
$(a^{\frac{1}{n}})^n = a^{\frac{1}{n} \cdot n} = a^{\frac{n}{n}} = a^1 = a$	$(\sqrt[n]{a})^n = a$	$a^{\frac{1}{n}} = \sqrt[n]{a}$

Why must x be nonnegative in the third row of the table? In the last row, if a were negative and n were even, then $\sqrt[n]{a}$ would not be a real number.

Definition A number raised to the $\frac{1}{n}$th power can be expressed as a radical with the index n if the radical is defined.

$$\forall x \in \mathrm{R}\ (x \geq 0)\ \ \forall n \in \mathrm{W}\ \ x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$\forall x \in \mathrm{R}\ \ \forall \text{ odd } n\ \ x^{\frac{1}{n}} = \sqrt[n]{x}$$

It is convenient to let $-x^{\frac{1}{n}}$ mean the same as $-(x^{\frac{1}{n}})$. Thus, $25^{\frac{1}{2}}$ means $\sqrt{25}$, but $-25^{\frac{1}{2}}$ means $-\sqrt{25}$. Of course, $(-25)^{\frac{1}{2}}$ is not a real number.

Negative fractional exponents may be interpreted in the same way as negative integral exponents. For example, the exponent $-\frac{1}{2}$ means 1 divided by the base with the positive exponent $\frac{1}{2}$.

$$16^{-\frac{1}{2}} = \frac{1}{16^{+\frac{1}{2}}} = \frac{1}{\sqrt{16}} = \frac{1}{4}$$

Now try these

— Study each equation. Is the equation true or is it false? If it is false, change it to make it true.

1. $16^{\frac{1}{2}} = 4$ **2.** $8^{\frac{1}{3}} = 2$ **3.** $(-125)^{\frac{1}{3}} = -5$

4. $100^{\frac{1}{2}} = 50$ **5.** $27^{\frac{1}{3}} = 9$ **6.** $36^{-\frac{1}{2}} = \frac{1}{36^{\frac{1}{2}}} = \frac{1}{6}$

7. $49^{-\frac{1}{2}} = -7$ **8.** $\sqrt[n]{97} = 97^{\frac{1}{n}}$ **9.** $(81^5)^{\frac{1}{5}} = 81$

Answers: **1.** True. **2.** True. **3.** True. **4.** False; $100^{\frac{1}{2}} = 10$. **5.** False; $27^{\frac{1}{3}} = 3$. **6.** True. **7.** False; $49^{-\frac{1}{2}} = \frac{1}{7}$. **8.** True. **9.** True.

Exercises

A — Rename each number without using exponents or radicals.

1. $9^{\frac{1}{2}}$ **2.** $125^{\frac{1}{3}}$ **3.** $16^{\frac{1}{4}}$ **4.** $(-27)^{\frac{1}{3}}$

5. $81^{\frac{1}{4}}$ **6.** $625^{\frac{1}{4}}$ **7.** $121^{\frac{1}{2}}$ **8.** $144^{\frac{1}{2}}$

9. $-36^{\frac{1}{2}}$ **10.** $-81^{\frac{1}{4}}$ **11.** $-(\frac{1}{4})^{\frac{1}{2}}$ **12.** $(\frac{1}{27})^{\frac{1}{3}}$

13. $-(\frac{1}{27})^{\frac{1}{3}}$ **14.** $(-\frac{1}{27})^{\frac{1}{3}}$ **15.** $\frac{1}{27}^{-\frac{1}{3}}$ **16.** $-(\frac{1}{27})^{-\frac{1}{3}}$

17. $(\frac{4}{9})^{\frac{1}{2}}$ **18.** $(-\frac{8}{125})^{\frac{1}{3}}$ **19.** $-(\frac{8}{125})^{\frac{1}{3}}$ **20.** $-25^{\frac{1}{2}}$

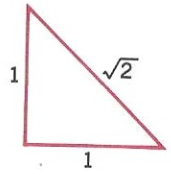
21. $-64^{\frac{1}{2}}$ **22.** 15^0 **23.** 9^{-2} **24.** 3^{-4}

— Use a radical to name each of the following.

25. $7^{\frac{1}{2}}$ **26.** $(-8)^{\frac{1}{3}}$ **27.** $16^{\frac{1}{4}}$ **28.** $93^{\frac{1}{n}}$

— Which triangles in Exercises 29–32 are right triangles? Do not trust the drawings. Instead, use the fact that only right triangles satisfy the Pythagorean theorem.

29.



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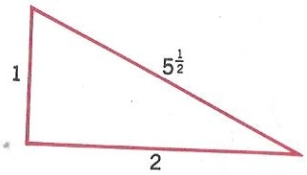

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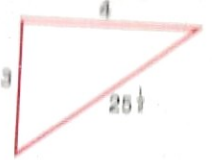

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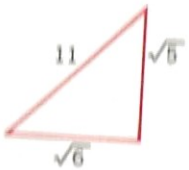


— What number is named when a is 8, b is 9, and c is 1?

33. $a^{\frac{1}{2}} - b^{\frac{1}{2}}$ **34.** $(a + c)^{\frac{1}{2}}$ **35.** $-a^{\frac{1}{2}}$ **36.** $-b^{\frac{1}{2}}$

37. $a^{-\frac{1}{3}}$ **38.** $b^{-\frac{1}{2}}$ **39.** $(bc)^{-\frac{1}{2}}$ **40.** $c^{-\frac{1}{2}}$

41. Make a table of the squares of each integer from 1 through 15. The members of the set $\{1, 4, 9, \cdots\}$ are called perfect squares because each is the square of an integer. Keep trying until you can name all the perfect squares from 1 through 225.

— For positive a, b, and c, if b is between a and c, then $\sqrt{b}$ is between $\sqrt{a}$ and $\sqrt{c}$. To approximate the irrational numbers named in the following sentences, replace x and y with consecutive integers to make true sentences.

42. $x < \sqrt{75} < y$ **43.** $x < \sqrt{42} < y$ **44.** $x < \sqrt{150} < y$

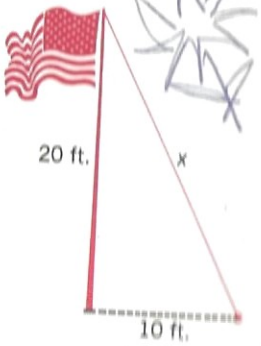
45. $x < \sqrt[3]{5} < y$ **46.** $x < \sqrt[5]{10} < y$ **47.** $x < \sqrt[4]{20} < y$

48. $x < \sqrt{200} < y$ **49.** $x < \sqrt[3]{-10} < y$ **50.** $x < \sqrt[3]{150} < y$

B **51.** The definition of rational exponents of the form $\frac{1}{n}$ is consistent with Theorems 6–5 and 6–6. Prove that it agrees with Theorems 6–1, 6–2, and 6–4 as well.

6–7 Approximations of Square Roots

The flagpole figure at the opening of the chapter is reproduced at the right. To install the guy wire whose length is x, you need to choose a rational number as an approximation for x. Imagine what a clerk in a hardware store would say if you asked for $\sqrt{500}$ feet of wire! The development that follows should help you find a way to give a rational approximation for $\sqrt{500}$.



As an approximation for $\sqrt{2}$, you might choose any of the rational numbers shown as approximations. The squares of the approximations are shown for comparison.

Approximation	Check
$\sqrt{2} \approx 1$	$1^2 = 1$
$\sqrt{2} \approx 2$	$2^2 = 4$
$\sqrt{2} \approx 1.5$	$1.5^2 = 2.25$
$\sqrt{2} \approx 1.4$	$1.4^2 = 1.96$
$\sqrt{2} \approx 1.41$	$1.41^2 = 1.9881$

From the table, you can see that $\sqrt{2}$ is somewhere between 1.41 and 1.5, but closer to 1.41. The approximations in the table were obtained by guessing. There is a systematic way to locate approximations to a square root that, after the first guess, always gives you a better approximation.

To find a rational approximation for a square root of a number, you make a guess as to the square root, divide the given number by the guess, average the divisor and quotient, divide by this mean, average again, and so on until you get the precision you want. In this way the divisor and quotient get closer and closer. If the divisor and the quotient were the same, each would be the square root. Explain.

EXAMPLE. Find a rational approximation for $\sqrt{42}$.

① You know $\sqrt{42}$ is between 6 and 7, so let your first guess be 6.5.

Divide 42 by 6.5.

$$\begin{array}{r} 6.4 \\ 6.5\overline{)42.0\,0} \\ 39\,0 \\ \hline 3\,0\,0 \\ 2\,6\,0 \\ \hline \end{array}$$

② Average 6.5 and 6.4. $\quad \frac{6.5 + 6.4}{2} = 6.45$

First approximation: $\sqrt{42} \approx 6.45$

Divide 42 by 6.45.

$$\begin{array}{r} 6.51 \\ 6.45\overline{)42.00\,00} \\ 38\,70 \\ \hline 3\,30\,0 \\ 3\,22\,5 \\ \hline 7\,50 \\ 6\,45 \\ \hline \end{array}$$

③ Average 6.45 and 6.51. $\quad \frac{6.45 + 6.51}{2} = 6.48$

Second approximation: $\sqrt{42} \approx 6.48$

Divide 42 by 6.48.

$$\begin{array}{r} 6.481 \\ 6.48\overline{)42.00\,000} \\ 38\,88 \\ \hline 3\,12\,0 \\ 2\,59\,2 \\ \hline 52\,80 \\ 51\,84 \\ \hline 960 \\ 648 \\ \hline \end{array}$$

④ Average 6.48 and 6.481. $\quad \frac{6.48 + 6.481}{2} = 6.4805$

Third approximation: $\sqrt{42} \approx 6.4805$

You can see that the divisor and dividend get closer and closer to the same number. You can continue this process as long as you wish and get a closer approximation for $\sqrt{42}$. But you can never find a rational number that is exactly $\sqrt{42}$ because $\sqrt{42}$ is irrational.

Now try these

1. $20^2 = \underline{?}$ $\quad 30^2 = \underline{?}$ Then is $20 < \sqrt{500} < 30$ true or false?

2. Divide 500 by 20. The divisor and the quotient tell you that $\sqrt{500}$ is between $\underline{?}$ and $\underline{?}$.

3. Average the divisor and quotient. Divide 500 by this number. Now you know that $\sqrt{500}$ is between $\underline{?}$ and $\underline{?}$.

4. To the nearest whole number, $\sqrt{500} \approx \underline{?}$.

Answers: **1.** 400; 900; true. **2.** 20; 25 **3.** 22.2; 22.5 **4.** 22

Exercises

A — What two consecutive integers could x and y represent in each sentence to make the sentence true?

1. $x < \sqrt{67} < y$ **2.** $x < \sqrt{47} < y$ **3.** $x < \sqrt{26} < y$

4. $x < \sqrt{101} < y$ **5.** $x < \sqrt{123} < y$ **6.** $x < \sqrt{150} < y$

7. $x < \sqrt{136} < y$ **8.** $x < \sqrt{46} < y$ **9.** $x < \sqrt{56} < y$

10. $x < \sqrt{13} < y$ **11.** $x < \sqrt{260} < y$

12. $x < \sqrt{901} < y$ **13.** $x < \sqrt{1001} < y$

14. $x < \sqrt{100{,}001} < y$ **15.** $x < \sqrt{8101} < y$

digits are in the numeral for the integral part of its square root?

17. What is the smallest positive integer whose square root is an integer named by two digits?

18. Use the fact that 10^2 equals 100 and 100^2 equals 10,000 to answer the following questions. For any positive number less than 10,000, what is the maximum number of digits in the numeral for the integral part of its square root? What is the least positive number whose square root is named by three digits?

— Which of the numbers below have square roots that are

a. less than 10?

b. greater than 10 but less than 100?

c. greater than 100 but less than 1000?

19. 64	**20.** 86	**21.** 144	**22.** 492
23. 4900	**24.** 7638	**25.** 12,100	**26.** 16,419
27. 16,900	**28.** 23,842	**29.** 250,000	**30.** 756,208

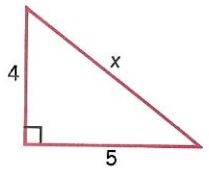
— Approximate each irrational number with a rational number to the nearest tenth.

31. $\sqrt{17}$	**32.** $\sqrt{12}$	**33.** $\sqrt{3}$	**34.** $\sqrt{130}$
35. $\sqrt{60}$	**36.** $90^{\frac{1}{2}}$	**37.** $450^{\frac{1}{2}}$	**38.** $800^{\frac{1}{2}}$

39. Find a rational approximation for $\sqrt{300}$ to the nearest hundredth.

— Find x to the nearest tenth.

40.



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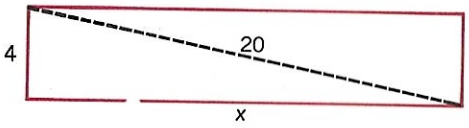
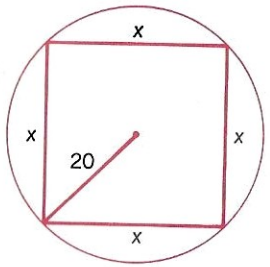


B **42.**



6–8 Principal Square Roots with Variables

If x is a real number, you know that x^2 is never negative. There are two real numbers that are the square root of x^2, of which $\sqrt{x^2}$ is the principal root because it is nonnegative.

If $x = 3$,	then	$\sqrt{x^2} = x$	because	$\sqrt{9} = 3$.
If $x = -3$,	then	$\sqrt{x^2} \neq x$	because	$\sqrt{9} \neq -3$.

When the square root of the square of a variable is involved, you can use the absolute value symbol to indicate a square root that will apply for both positive and negative replacements of the variable.

If $x = 3$,	then	$\sqrt{x^2} = \lvert x \rvert$	because	$\sqrt{9} = \lvert 3 \rvert$.
If $x = -3$,	then	$\sqrt{x^2} = \lvert x \rvert$	because	$\sqrt{9} = \lvert -3 \rvert$.

The examples below illustrate the way you can name square roots that involve variables.

a. $\sqrt{64a^2} = \lvert 8a \rvert$, or $8 \lvert a \rvert$

b. $(36b^2)^{\frac{1}{2}} = \lvert 6b \rvert$, or $6 \lvert b \rvert$

c. $\sqrt{49y^8} = \lvert 7y^4 \rvert$, or $7y^4$

d. $(81x^4y^6)^{\frac{1}{2}} = \lvert 9x^2y^3 \rvert$, or $9x^2 \lvert y^3 \rvert$

Why is it correct in **c** to rename $\lvert 7y^4 \rvert$ as $7y^4$? Does $7y^4$ always name a positive number? Explain **d.**

Now try these

— Rename each number without using the radical sign. The replacement set for each variable is the set of real numbers.

1. $\sqrt{16x^2}$ **2.** $\sqrt{100x^2y^4}$ **3.** $\sqrt{0.36x^6}$ **4.** $\sqrt{400x^{12}y^4}$

Answers: **1.** $\lvert 4x \rvert$, or $4 \lvert x \rvert$ **2.** $\lvert 10xy^2 \rvert$, or $10y^2 \lvert x \rvert$ **3.** $\lvert 0.6x^3 \rvert$, or $0.6 \lvert x^3 \rvert$ **4.** $\lvert 20x^6y^2 \rvert$, or $20x^6y^2$

Exercises

A — Rename each of the following numbers and expressions without using rational exponents or radical signs.

1. $\sqrt{4n^2}$	**2.** $(49b^2)^{\frac{1}{2}}$	**3.** $\sqrt{64x^4}$
4. $(36x^2)^{\frac{1}{2}}$	**5.** $\sqrt{100x^2}$	**6.** $(100y^2)^{\frac{1}{2}}$
7. $\sqrt{x^2y^2}$	**8.** $(x^2y^2z^4)^{\frac{1}{2}}$	**9.** $\sqrt{64a^2b^2}$
10. $\sqrt{\frac{4}{9}x^2}$	**11.** $\sqrt{\frac{1}{4}a^2}$	**12.** $\sqrt{\frac{1}{9}b^2}$
13. $(9a^4b^2)^{\frac{1}{2}}$	**14.** $(0.81y^2)^{\frac{1}{2}}$	**15.** $\sqrt{0.16t^4}$
16. $\sqrt{16x^{16}}$	**17.** $\sqrt{36x^{36}}$	**18.** $\sqrt{81a^4b^6}$
19. $(100x^8y^2)^{\frac{1}{2}}$	**20.** $(16a^8b^4)^{\frac{1}{2}}$	**21.** $\sqrt{100x^2y^6}$

22. $\sqrt{0.01a^2}$ 23. $(169a^4b^8c^6)^{\frac{1}{2}}$ 24. $(\frac{1}{25}r^8s^{12})^{\frac{1}{2}}$

25. $\sqrt{121x^6y^{12}}$ 26. $\sqrt{0.25a^6b^{10}}$ 27. $(225r^{10})^{\frac{1}{2}}$

28. $(1.44a^6b^8)^{\frac{1}{2}}$ 29. $(144a^{12})^{\frac{1}{2}}$ 30. $(9a^2b^6)^{\frac{1}{2}}$

31. $(81m^{10})^{\frac{1}{4}}$ 32. $(\frac{4}{25}x^8y^6)^{\frac{1}{2}}$ 33. $\sqrt{\frac{1}{9}c^{20}}$

34. $\sqrt{225x^{10}y^{10}}$ 35. $(400x^6y^4)^{\frac{1}{2}}$ 36. $\sqrt{900x^2y^8}$

37. $(4900x^{30}y^{60})^{\frac{1}{2}}$ 38. $\sqrt{36x^2y^{100}}$ 39. $\sqrt{100x^2y^{12}}$

B 40. $\sqrt{(x+y)^2}$ 41. $\sqrt{(2y-3)(2y-3)}$

42. $\sqrt{x^2+2xy+y^2}$ 43. $\sqrt{4y^2-12y+9}$

6–9 Simplifying Radicals

Give a reason why each of the following is true.

a. $\sqrt{36} = 6$ b. $\sqrt{4}\cdot\sqrt{9} = 2\cdot 3 = 6$

c. $\sqrt{36} = \sqrt{4\cdot 9}$ d. $\sqrt{4\cdot 9} = \sqrt{4}\cdot\sqrt{9}$

Sentences **a–d** show that $\sqrt{4\cdot 9}$ and $\sqrt{4}\cdot\sqrt{9}$ name the same number. It seems reasonable to ask if it is true in all cases that $\sqrt{ab}$ names the same number as $\sqrt{a}\cdot\sqrt{b}$ $(a, b \geq 0)$. To prove that this is indeed true, you use the definition of square root. The numerical example in the left column below may help you with the general proof shown in the right column.

Show that $\sqrt{2\cdot 3} = \sqrt{2}\cdot\sqrt{3}$.

The square of $\sqrt{2\cdot 3}$ is $2\cdot 3$.

$$(\sqrt{2}\cdot\sqrt{3})^2 = (\sqrt{2}\cdot\sqrt{3})(\sqrt{2}\cdot\sqrt{3}) = (\sqrt{2}\cdot\sqrt{2})(\sqrt{3}\cdot\sqrt{3}) = 2\cdot 3$$

Since $(\sqrt{2}\cdot\sqrt{3})^2$ is $2\cdot 3$, $\sqrt{2}\cdot\sqrt{3}$ is $\sqrt{2\cdot 3}$.

Prove that $\sqrt{ab} = \sqrt{a}\cdot\sqrt{b}$.

The square of $\sqrt{ab}$ is ab.

$$(\sqrt{a}\cdot\sqrt{b})^2 = (\sqrt{a}\cdot\sqrt{b})(\sqrt{a}\cdot\sqrt{b}) = (\sqrt{a}\cdot\sqrt{a})(\sqrt{b}\cdot\sqrt{b}) = ab$$

Since $(\sqrt{a}\cdot\sqrt{b})^2$ is ab, $\sqrt{a}\cdot\sqrt{b}$ is $\sqrt{ab}$.

This completes the proof of the following theorem.

Theorem 6–7 The square root of the product of two nonnegative numbers is the product of their square roots.

$$\forall a\in R\ (a \geq 0)\quad \forall b\in R\ (b \geq 0)\quad \sqrt{a\cdot b} = \sqrt{a}\cdot\sqrt{b}$$

Why is it necessary that a and b in Theorem 6–7 be nonnegative?

By an argument similar to the one for Theorem 6–7, a theorem for any root can be proved. It is stated below with the proof left as an exercise.

Theorem 6–8 The nth root of the product of two nonnegative numbers is the product of their nth roots.

$$\forall a\in R\ (a \geq 0)\quad \forall b\in R\ (b \geq 0)\quad \sqrt[n]{a\cdot b} = \sqrt[n]{a}\cdot\sqrt[n]{b}$$

Theorem 6–7 is useful when one factor of the radicand is a perfect square. The radical is said to be *simplified* when it has no perfect square as a factor.

Explain how the theorems of this section are used in simplifying the radicals and powers in **e–j**. Note that in **e–h**, one factor of each radicand, or base, is a perfect square.

e. $\sqrt{50} = \sqrt{25\cdot 2} = \sqrt{25}\cdot\sqrt{2} = 5\sqrt{2}$

f. $(48)^{\frac{1}{2}} = (16\cdot 3)^{\frac{1}{2}} = (16)^{\frac{1}{2}}(3)^{\frac{1}{2}} = 4(3)^{\frac{1}{2}}$, or $4\sqrt{3}$

g. $\sqrt{50x^3} = \sqrt{25x^2\cdot 2x} = \sqrt{25x^2}\cdot\sqrt{2x} = |5x|\sqrt{2x}$

h. $3\sqrt{8} = 3\sqrt{4\cdot 2} = 3\sqrt{4}\cdot\sqrt{2} = 3\cdot 2\cdot\sqrt{2} = 6\sqrt{2}$

i. $\sqrt[3]{54} = \sqrt[3]{27}\cdot\sqrt[3]{2} = 3\sqrt[3]{2}$

j. $\sqrt[4]{32} = \sqrt[4]{16}\cdot\sqrt[4]{2} = 2\sqrt[4]{2}$

Now try these

Express each number below as the product of a perfect square and another number. In Exercises 10–15, choose the greatest perfect square possible as one of the factors.

1. $50 = \underline{?}\cdot 2$ 2. $27 = \underline{?}\cdot 3$ 3. $200 = \underline{?}\cdot 2$

4. $24 = \underline{?}\cdot 6$ 5. $40 = \underline{?}\cdot 10$ 6. $4a^3 = 4a^2\cdot\underline{?}$

7. $8x^3y = 4x^2\cdot\underline{?}$ 8. $20a^3b^2 = 4a^2b^2\cdot\underline{?}$ 9. $27x^8y = \underline{?}\cdot 3y$

10. $75 = \underline{?}\cdot\underline{?}$ 11. $125 = \underline{?}\cdot\underline{?}$ 12. $288 = \underline{?}\cdot\underline{?}$

13. $338 = \underline{?}\cdot\underline{?}$ 14. $98 = \underline{?}\cdot\underline{?}$ 15. $300 = \underline{?}\cdot\underline{?}$

Simplify these radicals.

16. $\sqrt{300}$ 17. $\sqrt{32}$ 18. $\sqrt{98}$ 19. $\sqrt{20a^3b^2}$

Answers: 1. 25 2. 9 3. 100 4. 4 5. 4 6. a 7. $2xy$ 8. $5a$ 9. $9x^8$
10. $25\cdot 3$ 11. $25\cdot 5$ 12. $144\cdot 2$ 13. $169\cdot 2$ 14. $49\cdot 2$
15. $100\cdot 3$ 16. $10\sqrt{3}$ 17. $4\sqrt{2}$ 18. $7\sqrt{2}$ 19. $|2ab|\sqrt{5a}$

Exercises

A Simplify each of the following radicals or powers.

1. $\sqrt{8}$	**2.** $\sqrt{40}$	**3.** $\sqrt{20}$	**4.** $\sqrt{24}$
5. $12^{\frac{1}{2}}$	**6.** $63^{\frac{1}{2}}$	**7.** $\sqrt{75}$	**8.** $\sqrt{125}$
9. $\sqrt[3]{243}$	**10.** $5(96)^{\frac{1}{2}}$	**11.** $32^{\frac{1}{2}}$	**12.** $2(40)^{\frac{1}{2}}$
13. $242^{\frac{1}{2}}$	**14.** $\sqrt[6]{64}$	**15.** $\sqrt{196}$	**16.** $\sqrt[3]{-8}$
17. $4\sqrt{72}$	**18.** $(-27)^{\frac{1}{3}}$	**19.** $\sqrt{28}$	**20.** $3(18)^{\frac{1}{2}}$
21. $(36+49)^{\frac{1}{2}}$	**22.** $3\sqrt[3]{54}$	**23.** $54^{\frac{1}{3}}$	**24.** $\sqrt[6]{128}$
25. $(b^8)^{\frac{1}{4}}$	**26.** $\sqrt{b^5}$	**27.** $\sqrt{b^7}$	**28.** $(8a^2)^{\frac{1}{2}}$
29. $(8a^3)^{\frac{1}{3}}$	**30.** $\sqrt{27a^2}$	**31.** $(27b^3)^{\frac{1}{3}}$	**32.** $\sqrt{a^5}$
33. $\sqrt{a^4b^3}$	**34.** $(36m^3)^{\frac{1}{2}}$	**35.** $(a^2b^3)^{\frac{1}{2}}$	**36.** $\sqrt{a^3b^2}$
37. $\sqrt{20a^2}$	**38.** $\sqrt{20a^3}$	**39.** $\sqrt{20a^3b^2}$	**40.** $(a^4b)^{\frac{1}{2}}$

From the table on page 546, you can find the following approximations for $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$.

$$\sqrt{2} \approx 1.414 \qquad \sqrt{3} \approx 1.732 \qquad \sqrt{5} \approx 2.236$$

Use these three approximations to name the nonnegative square root of each number below to the nearest hundredth.

Example. 300

$$\sqrt{300} = \sqrt{100} \cdot \sqrt{3} = 10\sqrt{3} \approx 10(1.732) = 17.32$$

Hence, $\sqrt{300} \approx 17.32$.

41. 8	**42.** 12	**43.** 20	**44.** 45
45. 98	**46.** 50	**47.** 147	**48.** 320
49. 192	**50.** 216	**51.** 360	**52.** 200

B **53.** Prove Theorem 6–8.

6–10 Multiplication of Radical Expressions

You can use Theorem 6–7 and the symmetric property of equality to find the products of radicals with the same index. Is each of the following true?

a. $\sqrt{2} \cdot \sqrt{3} = \sqrt{2 \cdot 3} = \sqrt{6}$ **b.** $\sqrt{6} \cdot \sqrt{7} = \sqrt{6 \cdot 7} = \sqrt{42}$

c. $\sqrt[3]{2} \cdot \sqrt[3]{3} = \sqrt[3]{2 \cdot 3} = \sqrt[3]{6}$ **d.** $\sqrt[4]{5} \cdot \sqrt[4]{6} = \sqrt[4]{5 \cdot 6} = \sqrt[4]{30}$

If the index of two or more radicals is the same, you may multiply the radicands and use the same index for the radical representing the product.

You can also use the distributive postulate to find another name for a radical expression involving multiplication and addition.

$$2(\sqrt{2} + \sqrt{3}) = 2\sqrt{2} + 2\sqrt{3}$$

Sometimes the new name will be much simpler than the old one.

$$\begin{aligned} 2\sqrt{2}(3\sqrt{2} - 4\sqrt{3}) &= 2\sqrt{2}(3\sqrt{2}) + 2\sqrt{2}(-4\sqrt{3}) \\ &= 2 \cdot 3 \cdot \sqrt{2} \cdot \sqrt{2} + 2 \cdot (-4) \cdot \sqrt{2} \cdot \sqrt{3} \\ &= 6 \cdot 2 + (-8)\sqrt{6} \\ &= 12 - 8\sqrt{6} \end{aligned}$$

Generally, computation is less tedious if you first simplify the radical expression before applying the distributive postulate.

$$\begin{aligned} \sqrt{2}(3\sqrt{32} + 4\sqrt{45}) &= \sqrt{2}(3\sqrt{16} \cdot \sqrt{2} + 4\sqrt{9} \cdot \sqrt{5}) \\ &= \sqrt{2}(12\sqrt{2} + 12\sqrt{5}) \\ &= 2 \cdot 12 + 12\sqrt{10} \\ &= 24 + 12\sqrt{10} \end{aligned}$$

To simplify the indicated product of two binomial radical expressions, you apply the distributive postulate twice, as in the following examples. Notice that a and b must be nonnegative for the expressions to be defined.

EXAMPLE 1. Find the product of $(\sqrt{a} + 3)$ and $(\sqrt{b} + 2)$.

$$\begin{aligned} (\sqrt{a} + 3)(\sqrt{b} + 2) &= (\sqrt{a} + 3)(\sqrt{b}) + (\sqrt{a} + 3)(2) \\ &= \sqrt{ab} + 3\sqrt{b} + 2\sqrt{a} + 6 \end{aligned}$$

EXAMPLE 2. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$

$$\begin{aligned} (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) &= (\sqrt{a} + \sqrt{b})(\sqrt{a}) + (\sqrt{a} + \sqrt{b})(-\sqrt{b}) \\ &= (\sqrt{a})(\sqrt{a}) + (\sqrt{b})(\sqrt{a}) + (\sqrt{a})(-\sqrt{b}) \\ &\quad + (\sqrt{b})(-\sqrt{b}) \\ &= \sqrt{a^2} + \sqrt{ab} - \sqrt{ab} + (-\sqrt{b^2}) \end{aligned}$$

Since $+\sqrt{ab}$ and $-\sqrt{ab}$ are additive inverses, their sum is 0.

$$\begin{aligned} \sqrt{a^2} + \sqrt{ab} - \sqrt{ab} + (-\sqrt{b^2}) &= \sqrt{a^2} + 0 + (-\sqrt{b^2}) \\ &= a + 0 + (-b) \\ &= a - b \end{aligned}$$

A — Simplify each of the following indicated products. Exercises 13 and 19 are done for you.

1. $\sqrt{2} \cdot \sqrt{16}$
2. $\sqrt{3} \cdot \sqrt{7} \cdot \sqrt{2}$
3. $2\sqrt{3} \cdot 3\sqrt{6}$
4. $\sqrt[3]{3} \cdot \sqrt[3]{4}$
5. $\sqrt[3]{3} \cdot \sqrt[3]{9}$
6. $\sqrt[4]{18} \cdot \sqrt[4]{2}$
7. $\sqrt{27} \cdot \sqrt{2}$
8. $\sqrt{4} \cdot \sqrt{3}$
9. $\sqrt{a} \cdot \sqrt{b} \cdot \sqrt{c}$
10. $2\sqrt{18} \cdot 5\sqrt{2}$
11. $\sqrt{2} \cdot 3\sqrt[3]{8}$
12. $2\sqrt{3} \cdot \sqrt{6}$
13. $\sqrt{7x} \cdot \sqrt{2x} = |x|\sqrt{14}$
14. $\sqrt{x^2y} \cdot \sqrt{x}$
15. $\sqrt{15} \cdot \sqrt{5x}$
16. $\sqrt[4]{9x} \cdot \sqrt[4]{3x^2}$
17. $\sqrt[3]{9x} \cdot \sqrt[3]{3x^2}$
18. $\sqrt[3]{x^2} \cdot \sqrt[3]{xy}$
19. $3^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} = 12^{\frac{1}{2}}$
20. $2^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}$
21. $5(7^{\frac{1}{2}}) \cdot 3(5^{\frac{1}{2}})$
22. $5^{\frac{1}{3}} \cdot 2^{\frac{1}{3}}$
23. $5x^{\frac{1}{2}} \cdot 3x^{\frac{1}{2}}$
24. $2y^{\frac{1}{3}} \cdot 3y^{\frac{1}{3}}$

— Use the distributive postulate to rename each indicated product as an indicated sum.

25. $2(\sqrt{3} + \sqrt{6})$
26. $3(\sqrt{3} - 4\sqrt{3})$
27. $\sqrt{2}(\sqrt{3} + \sqrt{4})$
28. $2\sqrt{3}(3\sqrt{3} - 6\sqrt{9})$
29. $2(\sqrt{12} + \sqrt{72})$
30. $\sqrt{3}(5 + \sqrt{18})$

B — Simplify each of the following indicated products. The variables represent nonnegative real numbers.

31. $\sqrt{x} \cdot \sqrt{y} \cdot \sqrt{xy} \cdot \sqrt{x^2y}$
32. $\sqrt[3]{x} \cdot \sqrt[3]{y^2} \cdot \sqrt[3]{xy}$
33. $(\sqrt{a} + 2)(\sqrt{a} - 6)$
34. $(\sqrt{a} + \sqrt{c})(\sqrt{a} - \sqrt{c})$
35. $(3\sqrt{x} + 6)(2\sqrt{x} - 3)$
36. $(5\sqrt{a} - 3\sqrt{b})(2\sqrt{a} - \sqrt{b})$
37. $(3\sqrt{x} - \sqrt{y})(3\sqrt{x} + \sqrt{y})$
38. $(2 - 5\sqrt{x})(2 + 5\sqrt{x})$

6–11 Rationalizing the Denominator

Theorem 6–4 stated that when n is a positive integer,

$$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

is true. Have you also shown that the equation is true when n is a rational number with 1 as the numerator? This result may be written using radicals.

Theorem 6–9 The nth root of a number named by a fraction is the quotient formed when the nth root of the numerator is divided by the nth root of the denominator.

$$\forall a \in R\ (a \geq 0)\quad \forall b \in R\ (b > 0)\quad \forall n \in N\quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

When n is odd, it is not necessary to restrict a and b to nonnegative values. You can prove Theorem 6–9 by showing that the nth power of $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ is simply $\frac{a}{b}$.

Show in two ways that sentences **a–c** are true.

a. $\sqrt{\frac{9}{16}} = \frac{3}{4}$ **b.** $\sqrt[3]{\frac{8}{27}} = \frac{2}{3}$ **c.** $\sqrt[4]{\frac{16}{81}} = \frac{2}{3}$

In **d–f**, application of Theorem 6–9 may be used directly to give the simplest form of the radical.

d. $\sqrt{\frac{8}{16}} = \frac{\sqrt{8}}{\sqrt{16}} = \frac{\sqrt{4} \cdot \sqrt{2}}{\sqrt{16}} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$, or $\frac{1}{2}\sqrt{2}$

e. $\sqrt[3]{\frac{24}{27}} = \frac{\sqrt[3]{24}}{\sqrt[3]{27}} = \frac{\sqrt[3]{8} \cdot \sqrt[3]{3}}{\sqrt[3]{27}} = \frac{2\sqrt[3]{3}}{3}$, or $\frac{2}{3}\sqrt[3]{3}$

f. $\sqrt[4]{\frac{15}{81}} = \frac{\sqrt[4]{15}}{\sqrt[4]{81}} = \frac{\sqrt[4]{15}}{3}$, or $\frac{1}{3}\sqrt[4]{15}$

It is customary in mathematics to rename the number so that there is no radical in the denominator. If you have to compute an approximation of the number, it is easier to divide by a rational number than by an irrational number. Also, it is convenient to have a standard form for radicals.

The process of making the denominator a rational number is called **rationalizing the denominator.** You know that the numerator and denominator shown in a fraction may be multiplied by the same nonzero number without changing the value of the fraction. To rationalize the denominator, you choose the smallest multiple that makes the radicand of the denominator of the fraction a perfect square, cube, fourth power, or whatever power is equal to the index.

Explain the steps in each of **g–k.**

g. $\sqrt{\frac{9}{15}} = \frac{3}{\sqrt{15}} = \frac{3}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = \frac{3\sqrt{15}}{15} = \frac{\sqrt{15}}{5} = \frac{1}{5}\sqrt{15}$

h. $\sqrt[3]{\frac{1}{4}} = \frac{1}{\sqrt[3]{4}} = \frac{1}{\sqrt[3]{4}} \cdot \frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{1\sqrt[3]{2}}{\sqrt[3]{8}} = \frac{1\sqrt[3]{2}}{2} = \frac{1}{2}\sqrt[3]{2}$

to rationalize the denominator.

i. $\sqrt{\frac{8}{15}} = \frac{\sqrt{4}\cdot\sqrt{2}}{\sqrt{15}} = \frac{2\sqrt{2}}{\sqrt{15}} = \frac{2\sqrt{2}\cdot\sqrt{15}}{\sqrt{15}\cdot\sqrt{15}} = \frac{2\sqrt{30}}{15} = \frac{2}{15}\sqrt{30}$

j. $\sqrt[3]{\frac{3}{4}} = \frac{\sqrt[3]{3}}{\sqrt[3]{4}}\cdot\frac{\sqrt[3]{2}}{\sqrt[3]{2}} = \frac{\sqrt[3]{6}}{\sqrt[3]{8}} = \frac{\sqrt[3]{6}}{2} = \frac{1}{2}\sqrt[3]{6}$

k. $\sqrt[4]{\frac{14}{5}} = \frac{\sqrt[4]{14}}{\sqrt[4]{5}}\cdot\frac{\sqrt[4]{125}}{\sqrt[4]{125}} = \frac{\sqrt[4]{1750}}{5} = \frac{1}{5}\sqrt[4]{1750}$

To simplify the products of radical expressions in fraction form, you first find the product of the numerators and the product of the denominators. Then simplify the resulting expression.

EXAMPLE 1. $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{3}{5}} = \frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{5}} = \frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{5}\cdot\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{5}}$

$= \frac{\sqrt{2}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} = \frac{\sqrt{10}}{5}$, or $\frac{1}{5}\sqrt{10}$

EXAMPLE 2. $\sqrt[3]{\frac{1}{3}}\cdot\sqrt[3]{\frac{2}{3}} = \frac{\sqrt[3]{1}\cdot\sqrt[3]{2}}{\sqrt[3]{3}\cdot\sqrt[3]{3}} = \frac{\sqrt[3]{2}}{\sqrt[3]{9}} = \frac{\sqrt[3]{2}}{\sqrt[3]{9}}\cdot\frac{\sqrt[3]{3}}{\sqrt[3]{3}}$

$= \frac{\sqrt[3]{6}}{\sqrt[3]{27}} = \frac{\sqrt[3]{6}}{3}$, or $\frac{1}{3}\sqrt[3]{6}$

Exercises

A — Simplify each of the following expressions.

1. $\sqrt{\frac{1}{2}}$
2. $(\frac{1}{8})^{\frac{1}{2}}$
3. $(\frac{3}{5})^{\frac{1}{2}}$
4. $\sqrt{\frac{4}{5}}$
5. $\frac{\sqrt[3]{4}}{\sqrt[3]{2}}$
6. $(\frac{9}{16})^{\frac{1}{3}}$
7. $\frac{3}{(2)^{\frac{1}{4}}}$
8. $\frac{2\sqrt{3}}{5\sqrt{7}}$
9. $\frac{\sqrt[3]{5}}{\sqrt[3]{9}}$
10. $\frac{3}{3\sqrt[3]{5}}$
11. $\sqrt{\frac{2}{3}}$
12. $(\frac{45}{8})^{\frac{1}{2}}$
13. $\frac{5(3)^{\frac{1}{2}}}{3(18)^{\frac{1}{2}}}$
14. $\sqrt[4]{\frac{2}{32}}$
15. $\frac{\sqrt[3]{3}}{\sqrt[3]{5}}$
16. $(\frac{3}{25})^{\frac{1}{3}}$
17. $\sqrt{\frac{5}{11}}$
18. $\sqrt{\frac{8}{9}}$
19. $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{3}{8}}$
20. $\sqrt{\frac{5}{6}}\cdot\sqrt{\frac{3}{2}}$
21. $\sqrt[3]{\frac{2}{3}}\cdot\sqrt[3]{\frac{4}{9}}$
22. $\sqrt[4]{\frac{2}{3}}\cdot\sqrt[4]{\frac{8}{9}}$
23. $\sqrt{\frac{3}{7}}\cdot\sqrt{\frac{2}{3}}$
24. $\sqrt[3]{\frac{9}{2}}\cdot\sqrt[3]{\frac{3}{4}}$

B — Simplify each of the following expressions.

25. $\sqrt{\frac{c}{d}}$
26. $\sqrt[3]{\frac{2x}{3}}$
27. $\sqrt[3]{\frac{xy}{z^3}}$
28. $\sqrt[3]{\frac{3x^3}{8y^2}}$
29. $\sqrt[3]{\frac{3x}{y^2}}$
30. $\sqrt{\frac{xy}{13}}$

C — Simplify each of the following expressions.

31. $\sqrt{9-\frac{9}{4}}$
32. $\sqrt{3^2-(\frac{3}{2})^2}$
33. $\sqrt{25-\frac{25}{4}}$
34. $\sqrt{5^2-(\frac{5}{2})^2}$
35. $\sqrt{49-\frac{49}{4}}$
36. $\sqrt{7^2-(\frac{7}{2})^2}$
37. $4\sqrt{\frac{a^3}{8}}$
38. $\sqrt[m]{\frac{x^{m+1}}{y^{m-1}}}$
39. $\frac{a}{b}\sqrt{\frac{b}{a}}$
40. $\sqrt[4]{\frac{a}{b}}\cdot\sqrt[4]{\frac{a^2}{b^2}}$
41. $\sqrt[3]{\frac{4a^2}{2b}}\cdot\sqrt[3]{\frac{4a}{2b}}$
42. $\sqrt[4]{\frac{x^2}{y}}\cdot 3\sqrt[4]{\frac{x^2}{y^3}}$

6–12 Adding Radical Expressions

You know that you can use the distributive postulate to add the like terms $5x$ and $7x$.

$$5x + 7x = (5 + 7)\cdot x = 12x$$

You can use the distributive postulate also to add like radicals.

$$5\sqrt{y} + 7\sqrt{y} = (5 + 7)\cdot\sqrt{y} = 12\sqrt{y}$$

Note that the factor $\sqrt{y}$ appears in both terms. You must have the *same radical* if you are to use the distributive postulate in addition.

EXAMPLE 1. $3\sqrt{5} + 8\sqrt{5} - 2\sqrt{5} = (3 + 8 - 2)(\sqrt{5}) = 9\sqrt{5}$

EXAMPLE 2.

$$\begin{aligned} 3\sqrt{x} + 6\sqrt{y} - \sqrt{x} - 3\sqrt{y} &= 3\sqrt{x} - \sqrt{x} + 6\sqrt{y} - 3\sqrt{y} \\ &= (3 - 1)(\sqrt{x}) + (6 - 3)(\sqrt{y}) \\ &= 2\sqrt{x} + 3\sqrt{y} \end{aligned}$$

EXAMPLE 3.

$$\begin{aligned} \sqrt{48} + 6\sqrt{27} - 5\sqrt{12} &= \sqrt{16}\cdot\sqrt{3} + 6\sqrt{9}\cdot\sqrt{3} - 5\sqrt{4}\cdot\sqrt{3} \\ &= 4\sqrt{3} + 18\sqrt{3} - 10\sqrt{3} \\ &= (4 + 18 - 10)(\sqrt{3}) \\ &= 12\sqrt{3} \end{aligned}$$

first so that you can determine whether or not the distributive postulate can be used. The same is true in Example 4.

EXAMPLE 4. $\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{6}}=\frac{\sqrt{2}\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}+\frac{\sqrt{3}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}-\frac{\sqrt{1}\cdot\sqrt{6}}{\sqrt{6}\cdot\sqrt{6}}$

$$=\frac{\sqrt{6}}{3}+\frac{\sqrt{6}}{2}-\frac{\sqrt{6}}{6}$$
$$=\tfrac{1}{3}\sqrt{6}+\tfrac{1}{2}\sqrt{6}-\tfrac{1}{6}\sqrt{6}$$
$$=(\tfrac{1}{3}+\tfrac{1}{2}-\tfrac{1}{6})(\sqrt{6})$$
$$=\frac{(2+3-1)}{6}(\sqrt{6})$$
$$=\tfrac{4}{6}\sqrt{6},\text{ or }\tfrac{2}{3}\sqrt{6}$$

Exercises

A — Add or subtract as indicated. The replacements for the variables are nonnegative in Exercises 5 and 6, but can be any real numbers in the remaining exercises.

1. $3\sqrt{2}+2\sqrt{2}-4\sqrt{2}$
2. $5\sqrt{3}-2\sqrt{3}-6\sqrt{3}$
3. $6\sqrt{2}+8\sqrt{2}-9\sqrt{2}$
4. $-3\sqrt{3}+4\sqrt{3}-10\sqrt{3}$
5. $5\sqrt{x}+6\sqrt{x}-11\sqrt{x}$
6. $\sqrt{4x}-\sqrt{x}-\sqrt{36x}$
7. $\sqrt{12}-\sqrt{27}+\sqrt{48}$
8. $3\sqrt{18}-2\sqrt{32}-5\sqrt{50}$
9. $10\sqrt{8}-\sqrt{72}+3\sqrt{98}$
10. $3\sqrt[3]{8}-2\sqrt[3]{16}-6\sqrt[3]{54}$
11. $3\sqrt[3]{27}-\sqrt[3]{16}-6\sqrt[3]{54}$
12. $10\sqrt{8}-3\sqrt{98}+6\sqrt{72}$
13. $\sqrt{\frac{2}{3}}+5\sqrt{\frac{1}{6}}-3\sqrt{\frac{3}{2}}$
14. $\sqrt{\frac{36}{2}}-3\sqrt{\frac{49}{2}}-6\sqrt{\frac{25}{2}}$
15. $\sqrt{\frac{2}{3}}-3\sqrt{\frac{1}{6}}$
16. $\sqrt{\frac{2}{36}}-3\sqrt{\frac{2}{49}}-6\sqrt{\frac{2}{25}}$
17. $\sqrt[3]{x}-3\sqrt[3]{y}+5\sqrt[3]{x}+2\sqrt[3]{y}$
18. $2\sqrt{18}-3\sqrt{48}-3\sqrt{98}$
19. $-\sqrt{20}+\sqrt{45}$
20. $3\sqrt[4]{48}-2\sqrt[4]{162}$

B — Add or subtract as indicated.

21. $\sqrt[3]{54}-3\sqrt[3]{128}-6\sqrt[3]{16}$
22. $\sqrt[3]{-54}-3\sqrt[3]{-16}+4\sqrt[3]{128}$
23. $\sqrt{\frac{8}{9}}+2\sqrt{\frac{1}{2}}-3\sqrt{\frac{9}{8}}$
24. $\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}$
25. $\sqrt{\frac{y}{x}}-\sqrt{\frac{x}{y}}-3\sqrt{xy}$
26. $\sqrt[3]{xy^2}-3\sqrt[3]{x^4y^6}+\sqrt[3]{9x^2y^6}$

C 27. In Exercises 24 and 25, when would the radicals not represent real numbers?

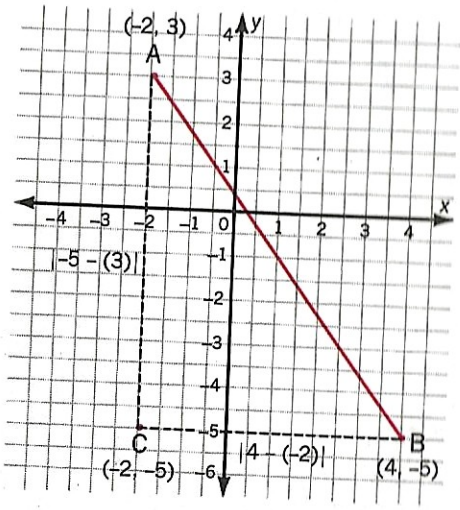
28. Perform this subtraction: $\sqrt{(a+b)^2c}-\sqrt{(a-b)^2c}$. What two things would you need to know about $(a+b)$ and $(a-b)$ to carry the subtraction further?

6–13 The Distance Formula

Look at the example below and try to discover a general method for finding the distance between any two points in the coordinate plane.

EXAMPLE 1. Find the distance from $A(-2, 3)$ to $B(4, -5)$.

The points are plotted and the line segment AB is drawn at the right. The dotted lines in color show the other two sides of right triangle ABC. Triangle ABC will be used to find $|\overline{AB}|$, the distance from A to B.



① Find the coordinates of C. They are the x coordinate of A and the y coordinate of B, or $(-2, -5)$.

② Find the length of $\overline{BC}$.

$$|\overline{BC}| = |4-(-2)|$$
$$= 6 \text{ units}$$

③ Find the length of $\overline{CA}$.

$$|\overline{CA}| = |-5-(+3)| = |-8|, \text{ or } 8 \text{ units}$$

④ Use the Pythagorean theorem to find $|\overline{AB}|$.

$$|\overline{AB}|^2 = |\overline{BC}|^2 + |\overline{CA}|^2$$

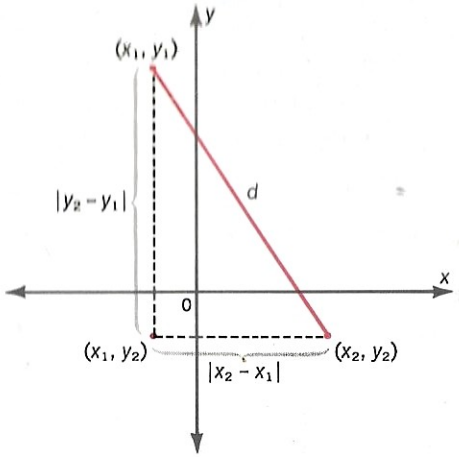
Let d represent $|\overline{AB}|$.

$$d^2 = 6^2 + 8^2$$
$$= 36 + 64$$
$$= 100$$

Hence, d is 10 or -10; but d must have the set of nonnegative numbers as its replacement set, since distance is nonnegative. The solution set for $d^2 = 6^2 + 8^2$ is $\{10\}$.

$$|\overline{AB}| = 10 \text{ units}$$

To obtain a **general distance formula**, choose any two points, (x_1, y_1) and (x_2, y_2), in the plane. One other vertex of a right triangle formed with these two points is (x_1, y_2). Why? The two legs of the right triangle have the measures $|x_2 - x_1|$ and $|y_2 - y_1|$. Then, by the Pythagorean theorem, the following sentences are true, where d is the distance between (x_1, y_1) and (x_2, y_2).



$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$
$$d = (|x_2 - x_1|^2 + |y_2 - y_1|^2)^{\frac{1}{2}}$$

The argument above proves the following theorem.

Theorem 6–10 The distance between any two points in the coordinate plane is found by (1) finding the difference between their x values and squaring it, (2) finding the difference between their y values and squaring it, and (3) adding the squares and then taking the nonnegative square root.

$$\forall P_1(x_1, y_1)\ \forall P_2(x_2, y_2)\quad |\overline{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Why was it permissible to drop the absolute-value signs for $x_2 - x_1$? for $y_2 - y_1$?

EXAMPLE 2. Find the distance between $P_1(7, 3)$ and $P_2(2, 5)$.

$$(x_2 - x_1) = (2 - 7) = -5; \qquad (x_2 - x_1)^2 = 25$$
$$(y_2 - y_1) = (5 - 3) = 2; \qquad (y_2 - y_1)^2 = 4$$

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$|P_1P_2| = \sqrt{25 + 4}$$
$$|P_1P_2| = \sqrt{29}$$

Now try these

— Use the Pythagorean theorem to find the distance between the points in each exercise.

1. (3, 7) and (8, −4) **2.** (−5, −4) and (8, −2)

— Use the distance formula to find the distance between the points in each exercise.

3. (1, 4) and (6, 9) **4.** (−3, 2) and (2, −7)

Answers: **1.** $\sqrt{146}$ **2.** $\sqrt{173}$ **3.** $\sqrt{50}$ or $5\sqrt{2}$ **4.** $\sqrt{106}$

Exercises

A — Find the distance between the points in each pair.

1. (−4, 5) and (5, −3) **2.** (0, 7) and (−5, −8)

3. $(a, 2)$ and $(a, -7)$ **4.** $(9, b)$ and $(-5, b)$

5. Find the lengths of the three sides of a triangle with vertices at (7, 3), (5, −3), and (0, 6).

6. In the pictures at the opening of the chapter, there is a cable stretched between $A(2, 5)$ and $B(5, 1)$. How long is the cable?

B **7.** The coordinates of each vertex of a quadrilateral are as follows.

$A(2, 7)$ $B(8, 3)$ $C(-4, -3)$ $D(5, -6)$

If $\overline{AB}$ and $\overline{CD}$ are opposite sides, find the lengths of all four sides. Is the quadrilateral a parallelogram? a rectangle? a square?

6–14 Solving Radical Equations

An equation such as $3 + \sqrt{x} = 6$ is a **radical equation.** To solve such an equation, you may add -3 to both sides and then square the sums to find a value for x. Solve $3 + \sqrt{x} = 6$.

Add −3 to each side. $\sqrt{x} = 3$
Square each side. $x = 9$

Check $3 + \sqrt{9} = 6$
$3 + 3 = 6$

Solution set: $\{9\}$

Add -4 to each side. $3\sqrt{2a-1} = 3$
Square each side. $9(2a-1) = 9$

Notice that the square of $3\sqrt{2a-1}$ is $3^2 \cdot (\sqrt{2a-1})^2$ just as $(3x)^2$ is $3^2 \cdot x^2$.

Use distributive postulate. $18a - 9 = 9$
Add 9 to each side. $18a = 18$
Divide by 18. $a = 1$

Check $3\sqrt{2(1)-1} + 4 = 7$
$3\sqrt{1} + 4 = 7$
$3 + 4 = 7$

Solution set: $\{1\}$

Squaring both sides of an equation sometimes gives a number that does not check in the original equation. When this is the case, the number is not a member of the solution set. For example, if you try to solve the equation $\sqrt{x} = -7$ by squaring both sides, you will get the equation $x = 49$. Checking gives $\sqrt{49} = -7$. But $\sqrt{49}$ is the non-negative root of 49, so the solution does not check. The sentence $\sqrt{49} = -7$ is not true. In fact, the equation $\sqrt{x} = -7$ has $\emptyset$ as the solution set. You must check all solutions in the original equation.

Exercises

A — Solve and check each equation. If an apparent solution does not check, give the solution set as $\emptyset$.

1. $3 + \sqrt{x-1} = 5$ **2.** $\sqrt{x} = 6$ **3.** $8 = \sqrt{x-9}$
4. $2\sqrt{x-5} = 7$ **5.** $3 - 2\sqrt{x+4} = 8$ **6.** $\sqrt{x} = -5$
7. $6 - \sqrt{x-5} = 2$ **8.** $5 + \sqrt{x} = 3$ **9.** $6 + \sqrt{x} = 13$
10. $8 + 2\sqrt{x} = 0$ **11.** $7 = 3\sqrt{x}$ **12.** $8\sqrt{x} = 4$
13. $\sqrt{\frac{n}{5}} = 2$ **14.** $7 = \frac{1}{\sqrt{y}}$ **15.** $8 = \sqrt{5r+1}$
16. $\sqrt{1} + 2x = \sqrt{5}$ **17.** $\sqrt{3y+2} = 2\sqrt{y}$ **18.** $+4 = \sqrt{7-2x}$
19. $3 = \frac{9}{\sqrt{y-1}}$ **20.** $\frac{\sqrt{x+3}}{2} = \sqrt{x}$ **21.** $3\sqrt{s} = 5\sqrt{3}$
22. $2\sqrt{x} = 5$ **23.** $5\sqrt{x} = 2$ **24.** $3 = \frac{5}{4\sqrt{x}}$
25. $\frac{\sqrt{11-2a}}{3} = \sqrt{a}$ **26.** $5 = \frac{15}{\sqrt{2a-3}}$ **27.** $\sqrt{7+3x} = -4$

B — Which number in each pair is the larger? You should be able to answer these without taking the square root.

28. $\sqrt{49}$, $\sqrt{50}$ **29.** $\sqrt{24}$, $\frac{\sqrt{63}}{2}$ **30.** $\frac{3\sqrt{7}}{2}$, $2\sqrt{6}$
31. $\frac{2\sqrt{5}}{3}$, $\frac{4\sqrt{3}}{3}$ **32.** $\frac{4\sqrt{3}}{3}$, $\frac{5\sqrt{2}}{3}$ **33.** $\frac{4\sqrt{3a}}{b}$, $\frac{5\sqrt{2a}}{b}$, $a, b > 0$

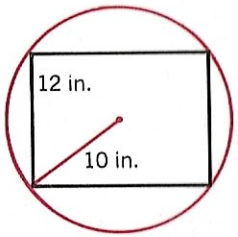
— Find the product.

34. $(2 + \sqrt{3})(2 - \sqrt{3})$ **35.** $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2})$
36. $\frac{4}{(\sqrt{2} - \sqrt{3})} \cdot \frac{5}{(\sqrt{2} + \sqrt{3})}$ **37.** $\frac{8}{\sqrt{7} - \sqrt{3}} \cdot \frac{2}{\sqrt{7} + \sqrt{3}}$

C — Write each of these fractions as a number with a denominator that is a rational number.

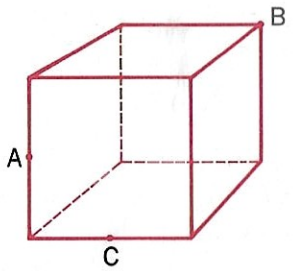
38. $\frac{2}{\sqrt{7}}$ **39.** $\frac{4}{\sqrt{2}+3}$ **40.** $\frac{5}{\sqrt{5}-\sqrt{2}}$
41. $\frac{\sqrt{6}+\sqrt{3}}{\sqrt{6}-\sqrt{3}}$ **42.** $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ **43.** $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

44. A rectangle of height 12 inches is inscribed in a circle with a radius of 10 inches. Find the area of the rectangle.



45. A man on a wharf 12 feet above the water pulls in a boat on the end of a rope. He had 20 feet of rope out and then pulled in 7 feet of rope. How far did he move the boat?

46. Each edge of the cube shown below measures 2 inches. Points A and C are the midpoints of the sides. What is the shortest distance from A to B to C, staying on the surface of the cube?



More Challenging Problems

1. Proclus, a Greek mathematician who lived about 460 B.C., showed that the sides of a right triangle, a, b, and c, can be expressed as $a = 2n + 1$; $b = 2n^2 + 2n$; and $c = 2n^2 + 2n + 1$, when n is any positive number. Use $a^2 + b^2 = c^2$ to show Proclus was correct.

Perform the indicated operations.

2. $(a^{n+1} + 5)(a^{n-1} + 5)$

3. $(x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1})$

4. $\left(\dfrac{9.3x^2y^2 + 1.5x^3y + 5.7x^2y^3}{-0.03x^2y}\right)(0.3x + 0.3y)$

5. $\dfrac{0.216x^3 - 0.333x^2y + 0.198xy^2 - 0.081y^3}{0.72x^2 - 0.39xy + 0.27y^2}$

6. $\dfrac{3x^4y^8 - x^3y^6 - 4x^2y^4 + 32xy^2 - 24}{x^2y^4 - 2xy^2 + 4}$

7. $(a^5 - 243) \div (a - 3)$

8. $(a^4 - 16) \div (a - 2)$

9. $\dfrac{1 - x - 3x^2 - x^5}{1 + 2x + x^2}$

10. $\dfrac{2 - 3a - 6a^2 + a^4}{1 - 2a - 2a^2 + a^3}$

11. $(a^{\frac{n}{2}} + b^{\frac{n}{2}} + c^{\frac{n}{2}})^2$

12. $(c - 5)^3 - (c + 5)^2$

13. $(x^4 + y^4)^2 - (x^4 - y^4)^2$

14. $(x^n + y^n)^2 - (x^n - y^n)^2$

15. What is the quotient if 2^{100} is divided by 2^6?

CHAPTER REVIEW

1. Know the meaning of and be able to use each of the following words or phrases. The number shown after each word or phrase indicates where it is introduced, in case you need to review.

exponent (*258*)
base (*258*)
power (*258*)
positive integral exponent (*258*)
zero exponent (*267*)
negative integral exponent (*267*)
expanded notation (*268*)
scientific notation (*268*)
square root (*270*)
radical (*271*)
radical sign (*271*)
radicand (*271*)
cube root (*271*)
index (*271*)
principal root (*271*)
*n*th root (*272*)
rational exponent, $\frac{1}{n}$ (*273*)
rationalizing the denominator (*285*)
distance formula (*290*)
radical equation (*291*)

Name without using exponents.

2. 5^3
3. $(-2)^4$
4. $(-1)^7$
5. $8^{\frac{1}{3}}$
6. 7^{-1}
7. 62^0
8. $(-1)^{\frac{1}{5}}$
9. $36^{-\frac{1}{2}}$
10. $(63)^{\frac{1}{2}}$

11. $(x^3) \cdot (x^4)$ means $(x \cdot x \cdot x) \cdot (x \cdot x \cdot x \cdot x)$ and equals $x^?$.

12. $\dfrac{a^7}{a^2}$ means $\dfrac{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a}{a \cdot a}$ and equals $a^?$.

13. **a.** $m^a \cdot m^b = m^?$ $(m \neq 0)$ **b.** $m^a \div m^b = m^?$ $(m \neq 0)$

14. The product of 234 and 234 is 54,756. What is the positive square root of 54,756? What is the negative square root of 54,756?

15. The principal square root of 207.36 is 14.4. What is the square of 14.4? of -14.4?

Replace x and y with consecutive integers to make true sentences.

16. $x < \sqrt{42} < y$
17. $x < -\sqrt{42} < y$
18. $x < \sqrt{78} < y$
19. $x < -\sqrt{78} < y$
20. $x < \sqrt{93} < y$
21. $x < -\sqrt{93} < y$

Which of the following numbers have positive square roots that are larger than 10 but smaller than 100?

22. 138
23. 56
24. 938
25. 759
26. 7595
27. 12,714

28. Which one of the numbers in the set $\{-6, -3, 0, 3, 6\}$ satisfies the equation $2 + \sqrt{x + 3} = 5$?

Find the square of each of the following.

29. 213
30. 42.6
31. 0.0721
32. $29x^2y$
33. $4ab^2c^3$
34. $7m^4n^5$

35. Find a decimal approximating $\sqrt{132}$ to the nearest hundredth.

Find the indicated root in simplest form.

36. $\sqrt{49x^2}$
37. $\sqrt{\dfrac{81x^2}{144a^2}}$
38. $c\sqrt{(a - 7)^2}$
39. $\sqrt{405}$
40. $[(3x + 2)^2]^{\frac{1}{2}}$
41. $\sqrt[4]{\dfrac{16x^2}{81y^2}}$
42. $\sqrt{\frac{9}{7}}$
43. $\sqrt{\frac{5}{12}}$
44. $(\frac{12}{13})^{\frac{1}{2}}$

45. Change $\dfrac{\sqrt{3}}{\sqrt{11}}$ to an equivalent fraction with a denominator that is a rational number.

46. How many feet of fence are needed for Bobby's garden if it is shaped like the diagram in Figure 1?

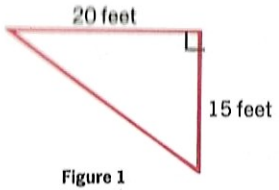


Figure 1

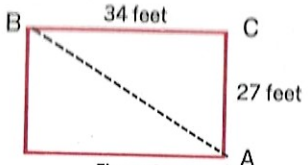


Figure 2

47. To the nearest tenth of a foot, how much farther is it from A to C to B than it is from A directly to B in Figure 2?

— Simplify and add.

48. $14\sqrt{3} - 2\sqrt{12}$ **49.** $\sqrt{32} - \sqrt{8}$ **50.** $5\sqrt{18ab} + 2\sqrt{72ab}$

51. $5\sqrt{7} - 2\sqrt{7}$ **52.** $\sqrt{20} + \sqrt{45}$ **53.** $a\sqrt{128} - 5a\sqrt{32}$

— Multiply or divide as indicated and leave the result in simplest radical form.

54. $\sqrt{3} \cdot \sqrt{27}$ **55.** $\sqrt{12} \div \sqrt{2}$

56. $4\sqrt{72} \div 3\sqrt{3}$ **57.** $4\sqrt{2} \cdot 3\sqrt{5}$

— Solve and check the following equations. If you get an apparent solution that does not check, indicate that $\varnothing$ is the solution set.

58. $3\sqrt{x} - 4 = 5$ **59.** $14 - \sqrt{x} = 15$

60. $8 = 2\sqrt{x}$ **61.** $1 = \frac{2}{\sqrt{x-4}}$

— Find the distance between the points in each pair.

62. (3, 6) and (2, 11) **63.** (−6, −5) and (−9, −8)

CHAPTER TEST

— Simplify each of the following.

1. $x^3 \cdot x^5$ **2.** $x^8 \div x^2$

3. $(x^5)^3$ **4.** $(-2^7)(2^4)$

5. $x(3)^2$ **6.** $\left(\frac{x^3}{y^4}\right)^3$

— Name each number without exponents.

7. $(\frac{4}{9})^{\frac{1}{2}}$ **8.** $(-1)^{\frac{1}{6}}$

9. $-(121^{\frac{1}{2}})$ **10.** $(-125)^{\frac{1}{3}}$

11. 5^{-2} **12.** 17^0

— Simplify each of the following expressions.

13. $\sqrt{3} \cdot \sqrt{4}$ **14.** $\sqrt{18}$

15. $\sqrt{a}(3 + \sqrt{6})$ **16.** $\sqrt[6]{64x^7}$

17. $\sqrt{\frac{16x}{9}}$ **18.** $\sqrt{\frac{1}{2}}$

19. $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$ **20.** $\sqrt{6} - 3\sqrt{6} + 2\sqrt{6}$

21. $\sqrt[3]{\frac{24}{54}}$ **22.** $3\sqrt{18} - 3\sqrt{48} - 3\sqrt{98}$

— Name each number with a rational denominator.

23. $\frac{2}{\sqrt{5}}$ **24.** $\frac{7}{3\sqrt{6}}$ **25.** $\sqrt{\frac{7}{12}}$ **26.** $\frac{4\sqrt{1}}{3\sqrt{14}}$

— Solve for x.

27. $\sqrt{1 + 2x} = \sqrt{7}$ **28.** $8 + \sqrt{x+2} = 17$

29. Find the distance from (3, −5) to (−7, 4).

— Simplify the following radicals.

30. $\sqrt[3]{\frac{75}{27y^3}}$ **31.** $\sqrt{\frac{15x^2}{8}}$ **32.** $\left(\frac{26}{8x}\right)^{\frac{1}{3}}$ **33.** $\sqrt[3]{\frac{8x^2}{3x^3}}$

— Use the distributive postulate to find the indicated products.

34. $(2\sqrt{x} - \sqrt{y})(\sqrt{x} + 3\sqrt{y})$

35. $(7 - 3\sqrt{x})(2 + 3\sqrt{x})$

36. $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$

CUMULATIVE REVIEW

— Give the letter of the response that best completes the sentence or answers the question.

1. If x is replaced by -3 and y by 4, what is the correct value of $6x - 5y$?

a. $(6 - 5)(-3 + 4)$ **b.** $(-18 - 5)4$

c. $(6 \cdot -3) - (5 \cdot 4)$ **d.** $(6 - 20)(-3)$

2. Which postulate for fractional numbers is illustrated by this sentence?

$$12(\tfrac{2}{3} + \tfrac{3}{4}) = 8 + 9$$

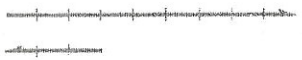
a. Associative postulate for addition.
b. Associative postulate for multiplication.
c. Distributive postulate.
d. Commutative postulate for multiplication.

a. The set {square, triangle, rectangle, parallelogram} is a subset of the set of quadrilaterals.
b. {natural numbers} $\subseteq$ {rational numbers}
c. {rational numbers} $\subseteq$ {integers}
d. {the center of a circle} $\subseteq$ {circumference of a circle}

4. Which of the following is equivalent to {9, 10, 25, 3}?
a. $\{\sqrt{100}, \sqrt{625}, \sqrt{81}, \sqrt{9}\}$
b. $\{5, 10, 15, 20, \cdots\}$
c. {21, 19, 15, 11, 7}
d. $\{\ \}$

5. Which second number is the negative of the first?
a. $^{+}7$ and $-(^{-}7)$ **b.** $|^{+}3|$ and $|^{-}3|$
c. $^{+}5$ and $|^{-}5|$ **d.** $-(^{-}6)$ and $+(^{-}6)$

6. Which sentence represents the difference of the two vectors shown here?



a. $x - y = -6$ **b.** $x - y = -12$
c. $x - y = +6$ **d.** $x - y = +12$

7. For which expression is $+8$ another name?
a. $(-7 + 2)(-3)$ **b.** $(-7 - 2)(+3)$
c. $-7(+2 - 3)$ **d.** $-2(-7 + 3)$

8. Which expression is equivalent to $-3m + \frac{n^2}{2} + 7m - \frac{3n^2}{2}$?
a. $+4m - \frac{n^2}{2}$ **b.** $-10m - n^2$ **c.** $4m + n^2$ **d.** $4m - n^2$

9. In one pair of equations, the second equation is obtained by using the multiplication postulate for equations. Which pair is it?
a. $3x - 2 = 5$ and $3x = 2 + 5$ **b.** $5x = -10$ and $x = -2$
c. $4x + 5 = 7$ and $4x = 7 - 5$ **d.** $8 - \frac{x}{2} = \frac{3x}{2}$ and $8 = 2x$

10. If the replacement set is {3, 5, 7, 9, 11}, what is the solution set of $\frac{2x}{2} - 2 = 5$?
a. $\varnothing$ **b.** {3, 5, 7}
c. {7} **d.** {3, 5, 7, 9, 11}

11. If the replacement set is the set of rational numbers, what is the solution set of the equation $\frac{3x}{4} - 7 = 2x + 9$?
a. $\{12\frac{4}{5}\}$ **b.** $\{-12\frac{4}{5}\}$ **c.** $\{6\frac{4}{5}\}$ **d.** $\{5\frac{9}{11}\}$

12. Which sentence is true?
a. $\forall x \in Q \quad x + 0 = 0$ **b.** $\forall x \in Q \quad \frac{2x}{3} = \frac{2}{3} + x$
c. $\exists x \in Q \quad x - 2 = 2 + x$ **d.** $\exists x \in Q \quad \frac{2x}{3} = \frac{2}{3} + x$

13. Sam's age is three times that of his youngest sister, Ann. In 7 years he will be only twice as old. If you are to find Sam's and Ann's present ages, which equation correctly represents this problem?
a. $3x - 7 = 2(x + 7)$ **b.** $3x + 7 = 2x + 7$
c. $3(x + 7) = 2(x + 7)$ **d.** $3x + 7 = 2(x + 7)$

14. If the replacement set is the set of real numbers, what is the solution set of $3x + 2 = 5$ *or* $3x + 2 > 5$?
a. $\{x : x > 1\}$ **b.** $\{x : x \geq 2\frac{1}{3}\}$
c. $\{x : x \geq 1\}$ **d.** $\{x : x = 2\frac{1}{2}\}$

15. Which ordered pair satisfies the equation $3x - 5y = 15$?
a. (5, 3) **b.** (−5, 0) **c.** (0, −3) **d.** (0, 3)

16. Which equation defines a linear function?
a. $3xy = 2$ **b.** $3(x + y) = 2$
c. $y(3x + 2) = 0$ **d.** $x(x + 2) = 3y$

17. If the graphs of the equations of a system are distinct lines with equal slopes, then the system is a(n)
a. simultaneous system. **b.** dependent system.
c. inconsistent system. **d.** consistent system.

18. Which equation has a graph with a y intercept of (0, −2) and a slope of $-\frac{2}{3}$?
a. $y = -\frac{2}{3}x + 2$ **b.** $y = -\frac{2}{3}x - 2$
c. $y = x - 2$ **d.** $y = \frac{2}{3}x + 2$

19. In which quadrant of the coordinate plane do the graphs of the equations of the following system intersect?

$$\begin{cases} 2x - y = 8 \\ x + 3y = -3 \end{cases}$$

a. First. **b.** Second. **c.** Third. **d.** Fourth.

20. What rational number is named by $(-\frac{1}{3})^{-2}$?

a. $\frac{1}{9}$ **b.** $-\frac{1}{9}$ **c.** 9 **d.** -9

21. Express 196.48 in scientific notation.

a. 1.9648×10^{-2} **b.** 1.9648×10^{2}
c. $19{,}648 \times 10^{2}$ **d.** 0.19648×10^{2}

22. What is the sum of $2\sqrt{8} + 3\sqrt{2}$?

a. $11\sqrt{2}$ **b.** $5\sqrt{10}$ **c.** $5\sqrt{2}$ **d.** $7\sqrt{2}$

23. A simple closed path formed by the union of three or more line segments is a

a. polygon. **b.** triangle. **c.** plane. **d.** line.

24. Which of the following is not used to define a relation?

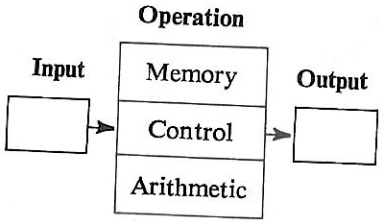
a. Graph. **b.** Equation.
c. Set of ordered pairs. **d.** The set of real numbers.

ENRICHMENT

A Computer Program for Linear Equations

In Chapter 5, you solved a linear equation by following a flow chart. In this chapter you will solve a linear equation by using the type of symbolic program that would be fed into a computer.

Basically, the parts of the computer are the Input, Operation, and Output; and Operation is subdivided into Memory, Control, and Arithmetic.



Into Input is fed information on punched cards or magnetic tape, and the information goes to Operation. Memory stores the information to be worked on. Arithmetic does the actual calculations. Control directs information between the other two sections of Operation and sends answers to Output, which prints out the answers on paper.

You and your pencil and paper will be the working parts of the computer, as well as the one who feeds it information. The information will be a list of code numerals, which would ordinarily be on separate punched cards. The information may be data or it may be instructions, but all information is in the form of a three-digit location numeral with a five-digit information numeral. If the three-digit numeral is 050 through 069 or 020 through 023, the information is data. Then the five-digit numeral that follows is just a decimal numeral. Thus,

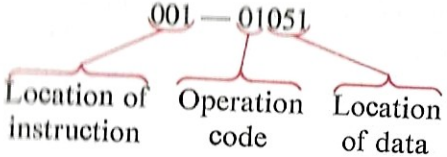
051 — 00006

means that 6 is stored in location 051.

If the three-digit numeral is anything but 050–069 or 020–023, then the five-digit numeral is read as a two-digit numeral with a three-digit numeral. The two-digit numeral tells the operation to be performed on the data stored in the location indicated by the three-digit numeral that follows it. The code numbers for the operations that you, the computer, will perform are the following.

Code number	Operation
00	Register all zeros in Arithmetic.
01	Register in Arithmetic the number in the location that follows.
02	Add to the number in Arithmetic the number from the location that follows.
03	Subtract from the number in Arithmetic the number in the location that follows.
04	Multiply the number in Arithmetic by the number in the location that follows.
05	Divide the number in Arithmetic by the number in the location that follows.
06	Replace the number in the location that follows by the number in Arithmetic. Clear Arithmetic for further operations.
07	Print out the number in the location that follows.
08	Compare the number in Arithmetic with the number in the location that follows: Answer "Yes" if the numbers are equal; answer "No" if they are unequal.
09	If answer to previous comparison is "No," go to the location that follows and continue program from there.
10	Take absolute value of the number in Arithmetic.
11	Take square root of the number in Arithmetic.
12	Stop.

Thus, the first instruction piece of information will be as follows.



001 means it is the first thing the computer will execute. 01 is read "Register in Arithmetic the number from the location that follows," which is 051. So you go to 051 and register whatever number it contains in Arithmetic. (*Register* means that you use the number and record it in a new location, but it is still in its old location for reuse.)

This program is going to allow you to solve from one to ten equations of the form $ax + b = 0$ $(a \neq 0, b \neq 0)$. Thus, the data would be $a_1x + b_1 = 0$, $a_2x + b_2 = 0, \cdots, a_{10}x + b_{10} = 0$. Each a and b must be stored in a separate location and they have been assigned the locations 050 to 069.

$$
\begin{aligned}
&050 — 0000a_1\\
&051 — 0000b_1\\
&052 — 0000a_2\\
&053 — 0000b_2\\
&\quad\vdots\\
&068 — 0000a_{10}\\
&069 — 0000b_{10}
\end{aligned}
$$

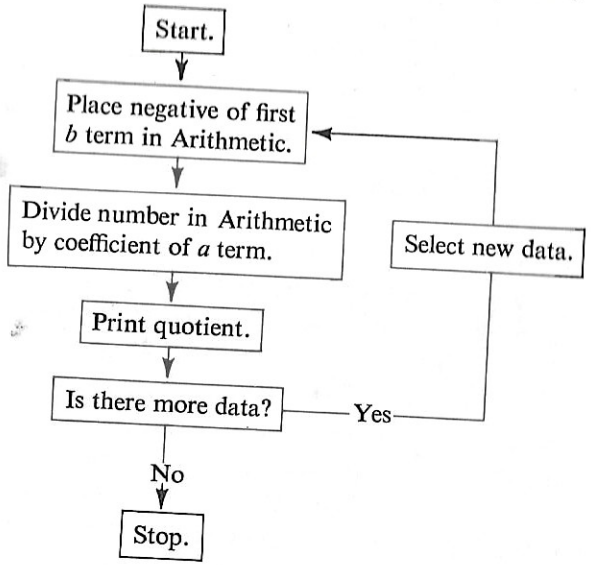
If fewer than ten equations are to be solved, 1 is inserted in place of a_n and 0 in place of b_n. This prevents division by 0 later in the program. Thus, if nine equations are given, the data would be the following.

$$
\begin{aligned}
&\vdots\\
&066 — 0000a_9\\
&067 — 0000b_9\\
&068 — 00001\\
&069 — 00000
\end{aligned}
$$

Before a program is written, a flow chart is drawn up as an outline of the steps to be covered in the program. This flow chart covers the steps needed for solving one equation, and then instructs the user to go back for new data and redo the same steps to solve the next equation.

This "going back" is known as a **loop.** A loop is an instruction to skip backward or forward over certain steps under certain conditions and is very useful in programming.

Flow Chart for Solving $ax + b = 0$ $(a \neq 0, b \neq 0)$



From the flow chart, the following step-by-step program is developed. As you read through the instructions, you will notice numbers and letters in parentheses. These are the results of solving the equation $a_1x + b_1 = 0$ and are, therefore, the general form of how the program works. The bracketed statements are further comments to clarify the procedure.

Location		Data or Instructions	Interpretation of Code Numeral
050	—	$0000a_1$	Put first x coefficient in 050.
051	—	$0000b_1$	Put first constant in 051.
.		.	.
.		.	.
.		.	.
068	—	$0000a_{10}$	Put tenth x coefficient in 068.
069	—	$0000b_{10}$	Put tenth constant in 069.
000	—	00000	Place all zeros in Arithmetic.
001	—	01051	Register number in 051 (b_1) in Arithmetic.
002	—	05050	Divide number in Arithmetic (b_1) by number in 050 (a_1).

(Continued on next page.)

003	—	04020	Multiply number in Arithmetic $\left(\frac{b_1}{a_1}\right)$ by number in 020 (−1).
004	—	06021	Move number in Arithmetic $\left(-\frac{b_1}{a_1}\right)$ to storage at 021. Clear Arithmetic for next operation.
005	—	07021	Print number in 021 $\left(-\frac{b_1}{a_1}\right)$. [At this point in the program, the first equation is now solved. Instructions 006 through 014 prepare the computer to go back to 000 and start again, this time solving the second equation.]
006	—	01001	Register in Arithmetic the number in 001 (01051).
007	—	02022	Add to the number in Arithmetic (01051) the number in 022 (00002).
008	—	06001	Replace number in 001 (01051) by number now in Arithmetic (01053). Clear Arithmetic for next operation.
009	—	01002	Register in Arithmetic the number in 002 (05050).
010	—	02022	Add to number in Arithmetic (05050) the number in 022 (00002).
011	—	06002	Replace number in 002 (05050) by number now in Arithmetic (05052). Clear Arithmetic for next operation.
012	—	01023	Register in Arithmetic the number in 023 (05070).
013	—	08002	Compare number in Arithmetic (05070) with number in 002 (05052). Answer "Yes" or "No." (No)
014	—	09000	If answer to 013 is "No" (It is.), return to 000 and rerun. [If answer to 013 is "Yes," the program proceeds.]
015	—	12000	Stop.
020	—	0000$\bar{1}$	Put −1 in 020. [−1 is symbolized by $\bar{1}$.]
021	—	00000	Put all zeros in 021. [To be used for storage.]
022	—	00002	Put 2 in 022.
023	—	05070	Put 05070 in 023.

All the information is fed into the computer. As it is initially put in, the computer just "reads" the information and stores the data and instructions in the appropriate locations. Then the computer returns to the beginning of the program and proceeds through it in *strict ascending order*. Thus, it makes no difference that 050–069 comes before 000, 001, · · · . The computer will begin the reading and execution of the program at 000, and will solve the equations one by one.

Now you, the programmer and the computer, can proceed to the exercises and solve the equations given.

Exercises

— Solve the following equations by using the program. Your first step should be to label a sheet of paper "Memory" and make a list of locations 050–069 with the proper data stored. Also label a sheet of paper "Arithmetic" and one "Output." Be careful to do each step and to do it correctly.

1. $2x + 4 = 0,\ 3x - 9 = 0$
2. $2x + 6 = 0,\ 3x + 12 = 0,\ 5x - 10 = 0,\ 3x + 6 = 0$
3. $x - 1 = 0,\ 2x + 7 = 0,\ -x - 9 = 0,\ 5x - 25 = 0,\ 3x + 21 = 0,\ -\frac{1}{2}x - 4 = 0,\ cx + e = 0,\ -5x + 1 = 0,\ -7x + 3 = 0,\ 8x + 24 = 0$

THE GROWTH OF MATHEMATICS

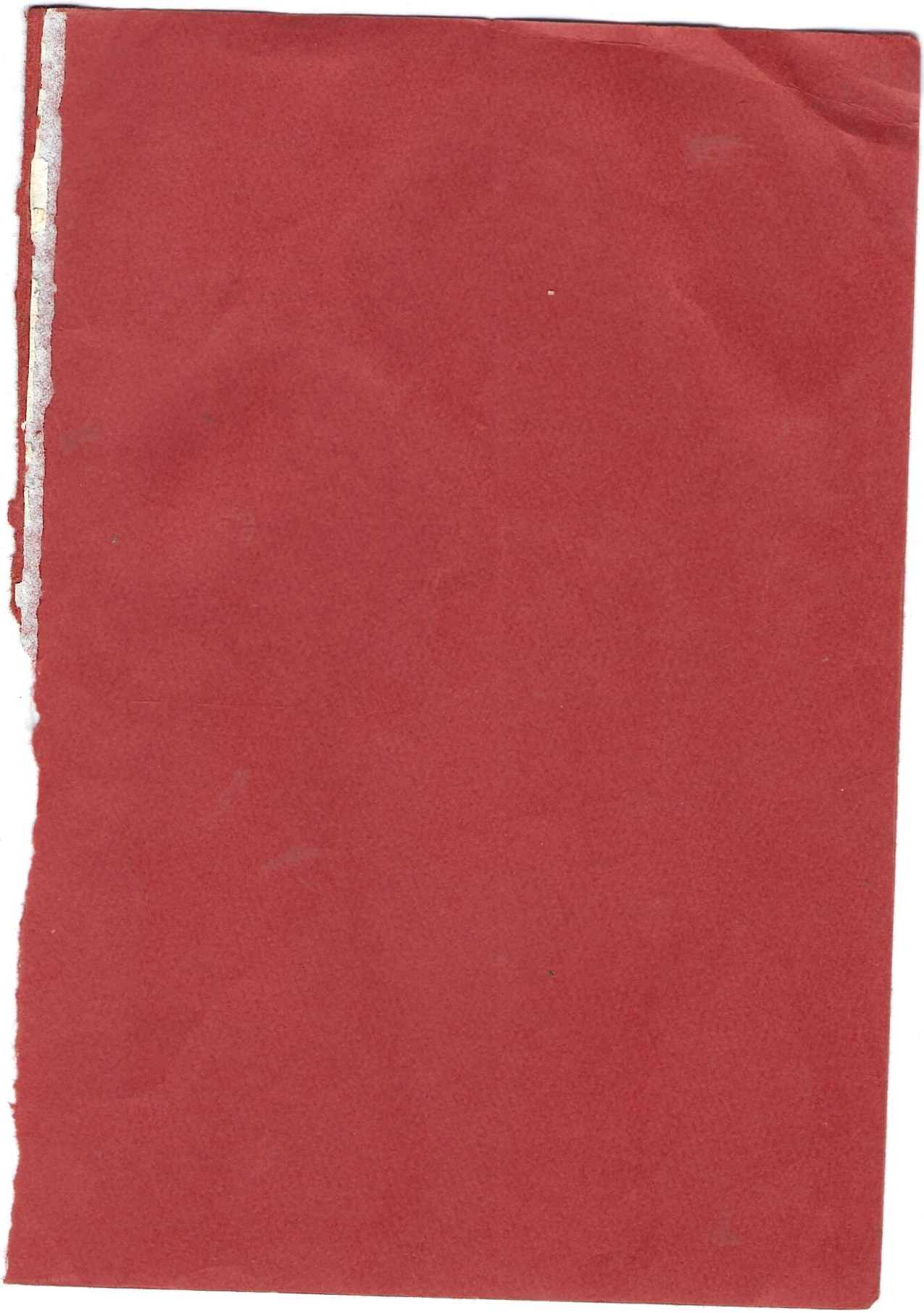
Archimedes

Archimedes (287?–212 B.C.) spent most of his life in the Greek city of Syracuse on the island of Sicily and is reputed to be *the* greatest mathematician of ancient times. Like Plato and others, he held applied mathematics in contempt. But he was a mechanical genius and often began with mechanical observations and then proceeded to his rigorous mathematical proofs. His discovery of the principles of floating bodies, that is, hydrostatics, is said to have been discovered as he was taking a bath. He leaped out of the bath and ran through the city crying "Eureka!" ("I have found it!")

He was familiar with the method of exhaustion and used it to *discover* the method for finding the areas of various regions, rather than to prove results as his predecessors had done; thus, he anticipated the calculus by two thousand years. Another major discovery was the law of the lever.

As a favor to his friend King Hiero, he built war machines that completely stunned a Roman army and put off the attack of Syracuse. But in another attack Archimedes was killed. On his tomb was inscribed, at his wishes, a sphere inside a circular cylinder, commemorating his discovery of the method for finding the volume of a sphere and its surface area.

TO BE
CONTINUED



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